6 Dynamic Optimization in Continuous Time

6.1 Dynamic programming in continuous time

Consider the problem

$$\max_{c(t)_0^T} \int_0^T e^{-rt} u(k, c, t)\, dt \tag{1}$$

$$s.t.\, \dot{k} = f(k, c, t) \tag{2}$$

$$k(0) = \underline{k}, \tag{3}$$

with

$$k(T) = \bar{k} \text{ (case 1), or} \tag{4}$$

$$k(T) \text{ free (case 2), or} \tag{5}$$

$$k(T) \geq \bar{k} \text{ (case 3).} \tag{6}$$

Thinking of the integral in the maximand as a sum of rectangles with base dt and height $e^{-rt}u(k, c, t)$, we can approximate the problem with a discrete time problem. Noting that for a small time interval dt, $k(t + dt) = k(t) + f(k, c, t)\, dt$, we can write the current value Bellman equation

$$V(k, t) = \max_c \left\{ u(k, c, t)\, dt + e^{-rdt} V(k + f(k, c, t)\, dt, t + dt) \right\}. \tag{7}$$

We can the make Taylor approximations;

$$V(k, t) = \max_c \{ u(k, c, t)\, dt \tag{8}$$

$$+ (1 - rdt)(V(k, t) + V_k(k, t) f(k, c, t)\, dt + V_t(k, t)\, dt) \} \tag{9}$$

Subtracting $(1 - rdt) V(k, t)$ from both sides, dividing by dt and then letting dt go to zero yields,

$$rV(k, t) = \max_c \{ u(k, c, t) + V_k(k, t) f(k, c, t) + V_t(k, t) \} \tag{10}$$

$$= \max_c \left\{ u(k, c, t) + \frac{dV(k, t)}{dt} \right\}, \tag{11}$$

where we note that $dV(k, t)/dt$ is the total time derivative of the value function. Sometimes, this equation is referred to as an asset pricing equation – and interpreted as follows; if an asset is correctly valued (not providing arbitrage opportunities) the opportunity cost of holding it (the LHS of (10)) equals the (optimal) sum of the immediate pay-off or dividend (the first term of (11)) and the capital gain (the second term of (11)).

Sometimes we can use the guess and verify technique to solve for the value function if the problem is autonomous. An alternative is to use *Optimal Control* and *Pontryagin's maximum principle.*

6.2 Optimal Control

Consider the problem in (1) with the continuous time Bellman equation (10). Noting that since the term $V_t(k,t)$ is independent of c, we can rewrite (10)

$$rV(k,t) - V_t(k,t) = \max_c \{u(k,c,t) + V_k(k,t) f(k,c,t)\} \tag{12}$$

Defining the *co-state* or current shadow value variable as the derivative of the value function w.r.t. k, along its optimal path k^*,

$$\lambda(t) \equiv V_k(k^*,t). \tag{13}$$

We see that a necessary condition for $c^*(t)$ to be optimal is that it is given by

$$c^*(t) = \arg\max_c \{u(k,c,t) + \lambda(t) f(k,c,t)\}. \tag{14}$$

Now, we need to pin down $\lambda(t)$. For, this purpose we analyze how λ develops over time by deriving a differential equation for $\lambda(t)$.Taking derivatives w.r.t. k of the identity (12), we get

$$rV_k(k,t) - V_{kt}(k,t) = u_k(k,c^*,t) + V_{kk}(k,t) f(k,c^*,t) \tag{15}$$
$$+V_k(k,t) f_k(k,c^*,t), \tag{16}$$
$$r\lambda(t) - \left(V_{tk}(k,t) + V_{kk}(k,t)\dot{k}\right) = u_k(k,c^*,t) + \lambda(t) f_k(k,c^*,t), \tag{17}$$
$$r\lambda(t) - \dot{\lambda}(t) = u_k(k,c^*,t) + \lambda(t) f_k(k,c^*,t), \tag{18}$$

where in second equation, we changed the order of diffentiation, using the fact that $\frac{\partial}{\partial k}\frac{\partial V(k,t)}{\partial t} = \frac{\partial}{\partial t}\frac{\partial V(k,t)}{\partial k}$.

Equation (14) and (18) form the basis of Pontryagin's maximum principle. According to this, we derive necessary (and sometimes sufficient) conditions for an optimal control. We do this without explicitly solving for the value function. We first define the *current value Hamiltoninan.* This is the sum of the instantaneous payoff and the product of costate(s) and the function determining the law-of-motion of the state variable;

$$\mathcal{H}(k,c,\lambda,t) \equiv u(k,c,t) + \lambda(t) f(k,c,t). \tag{19}$$

Note that the Hamiltonian has the same interpretation as the RHS of the Bellman equation without the max-operator. In words, it is the sum of the flow of current pay-off and the generation of future pay-off. In the Bellman equation, we use the value function to measure future payoffs while the Hamiltonian use the shadow-value λ.

According to Pontryagin maximum principle, the optimal control $c^*(t)$ maximizes the Hamiltonian at each instant, the co-state (or shadow value) satisfies the differential equation, $r\lambda(t) - \dot{\lambda}(t) = \mathcal{H}_k(k, c, \lambda, t)$. These necessary conditions will provide differential equations which we need to solve. Typically we have one initial condition for each state variable. But we need more information to solve the system since we also have the control variable(s). In case (1) of (4), we have the necessary additional information. In case (2), it must be that the shadow value of the state variable approach zero as $t \to T$. This is called, *the transversality condition.* In case (3), either the inequality is slack, in which case $\lambda(T) = 0$, or it binds, giving the necessary additional info in both cases. We can the summarize:

Result 17 *According to the* Pontryagin's maximum principle
1. An optimal control $c^*(t)$, *satisfies*

$$c^*(t) = \arg\max_c \mathcal{H}(k, c, \lambda, t), \tag{20}$$

2. where the current co-state or shadow value $\lambda(t)$, *is a continuous function of time, satisfying the differential equation.*

$$r\lambda(t) - \dot{\lambda}(t) = \mathcal{H}_k(k, c, \lambda, t), \tag{21}$$

except at points in time where c *is discontinuous and with*
3. end-condition(s) provided by

$$k(T) = \bar{k} \text{ (case 1) or} \tag{22}$$

$$\lambda(T) = 0 \text{ if } k(T) \text{ is free(case 2), or} \tag{23}$$

$$\lambda(T) \geq 0, \text{ and } \lambda(T)\left(k(T) - \bar{k}\right) = 0 \text{ if } k(T) \geq \bar{k} \text{(case (3).} \tag{24}$$

In addition, the path for $k(t)$ *must satisfy the initial condition,* $k(0) = \underline{k}$, *and* $\dot{k}(t) = \mathcal{H}_\lambda(k, c, \lambda, t) = f(k, c, t)$.

6.2.1 The consumption problem

As an example, consider the problem standard consumption-savings (Ramsey) problem

$$\max_{c(t)_0^T} \int_0^T e^{-rt} u(c)\, dt \tag{25}$$

$$s.t.\, \dot{k} = f(k) - c \tag{26}$$

$$k(0) = \underline{k}, \tag{27}$$

$$k(T) \geq 0. \tag{28}$$

with u and f increasing and concave. The current value Hamiltonian is

$$\mathcal{H}(k,c,\lambda,t) = u(c) + \lambda(t)(f(k) - c). \tag{29}$$

Thus, $c^*(t) = \arg\max_c u(c) + \lambda(t)(f(k) - c)$ which is interior, implying

$$\mathcal{H}_c(k,c,\lambda,t) = 0. \tag{30}$$

$$\rightarrow u'(c^*) = \lambda. \tag{31}$$

Furthermore, from the second condition

$$\mathcal{H}_k(k,c,\lambda,t) = \lambda f'(k) = r\lambda - \dot{\lambda}. \tag{32}$$

Taking time-derivatives of $u'(c^*) = \lambda$ and substituting into (32) we get

$$u''(c^*)\dot{c}^* = \dot{\lambda}, \tag{33}$$

$$u'(c^*) f'(k) = ru'(c^*) - u''(c^*)\dot{c}^*, \tag{34}$$

$$\rightarrow \dot{c}^* = \frac{u'(c^*)}{-u''(c^*)}(f'(k) - r), \tag{35}$$

which is the Euler equation we have seen before. To analyze the behavior of this system, we can use draw the phase-diagram as in section 3.5.

To get a closed form solution, i.e., an expression for the endogenous variables in terms of only the exogenous ones, we must specify the utility and investment functions. Considering first the utility functions, we have two important special cases. First, CARA utility,

$$u(c) = -\frac{e^{-\gamma c}}{\gamma}, \tag{36}$$

$$\rightarrow u'(c) = e^{-\gamma c}, u''(c) = -\gamma e^{-\gamma c} \tag{37}$$

in which case we get

$$\dot{c}^* = \frac{1}{\gamma}(f'(k) - r), \tag{38}$$

i.e., consumption growth is a linear function in the difference between the marginal return on savings and the subjective discount rate. The other case is CRRA,

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \tag{39}$$

$$\rightarrow u'(c) = c^{-\sigma}, u''(c) = -\sigma c^{-\sigma-1}, \tag{40}$$

yielding

$$\dot{c}^* = \frac{c^{-\sigma}}{\sigma c^{-\sigma-1}} \left(f'(k) - r\right) = \frac{c}{\sigma} \left(f'(k) - r\right) \tag{41}$$

$$\frac{\dot{c}^*}{c} = \frac{1}{\sigma} \left(f'(k) - r\right), \tag{42}$$

i.e., consumption growth *rate* is a linear function in $f'(k) - r$. The sensitivity is given by $1/\sigma$, which we call the intertemporal elasticity of substitution. Here, we also have that σ is the constant of relative riskaversion.

What about the transversality condition? In this case, we know $u'(c^*(T)) = \lambda(T)$. So since utility in the examples is unbounded, i.e., $u'(c) > 0$ for all finite c, $\lambda(T)$ cannot be 0, instead $k(T)$ is zero. In other words, whenever consumption has a value at T, the lower bound on k should bind and nothing should be left.

Let us complete the example by assuming, for simplicity, a linear (Romer type) production function $f(k) = Ak$. In the CRRA case, we get the linear system

$$\dot{c}^* = \frac{A - r}{\sigma} c \tag{43}$$

$$\dot{k} = -c + Ak \tag{44}$$

$$\begin{bmatrix} \dot{c}^* \\ \dot{k} \end{bmatrix} = \begin{bmatrix} \frac{A-r}{\sigma} & 0 \\ -1 & A \end{bmatrix} \begin{bmatrix} c^* \\ k \end{bmatrix}. \tag{45}$$

This system has roots A and $\frac{A-r}{\sigma}$ and the matrix of eigenvectors is

$$\begin{bmatrix} 0 & \frac{r+A(\sigma-1)}{\sigma} \\ 1 & 1 \end{bmatrix} \equiv \mathbf{B}^{-1}. \tag{46}$$

Consequently, the solution is of the form

$$\begin{bmatrix} \dot{c}^*(t) \\ \dot{k}(t) \end{bmatrix} = \mathbf{B}^{-1} \begin{bmatrix} e^{At} & 0 \\ 0 & e^{\frac{A-r}{\sigma}t} \end{bmatrix} \begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix} \tag{47}$$

$$= \begin{bmatrix} 0 & \frac{r+A(\sigma-1)}{\sigma} e^{\frac{A-r}{\sigma}t} \\ e^{At} & e^{\frac{A-r}{\sigma}t} \end{bmatrix} \begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix}, \tag{48}$$

where κ_1 and κ_2 are two integration constants. We solve for the latter by using $k(0) = \underline{k}$ and $k(T) = 0$.

$$\begin{aligned} k(0) &= \underline{k} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix} && (49) \\ &= \kappa_1 + \kappa_2 && (50) \\ k(T) &= 0 = \begin{bmatrix} e^T & e^{\frac{A-r}{\sigma}T} \end{bmatrix} \begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix} && (51) \\ &\rightarrow \kappa_1 = \frac{e^{\frac{A-r-\sigma}{\sigma}T}}{e^{\frac{A-r-\sigma}{\sigma}T} - 1}\underline{k}, \kappa_2 = \frac{1}{1 - e^{\frac{A-r-\sigma}{\sigma}T}}\underline{k}. && (52) \end{aligned}$$

We can now, for example, evaluate

$$\begin{aligned} \dot{c}^*(t) &= \frac{r + A(\sigma - 1)}{\sigma} e^{\frac{A-r}{\sigma}t} \frac{1}{1 - e^{\frac{A-r-\sigma}{\sigma}T}}\underline{k}, && (53) \\ \dot{c}^*(0) &= \frac{r + A(\sigma - 1)}{\sigma} \frac{1}{1 - e^{\frac{A-r-\sigma}{\sigma}T}}\underline{k}. && (54) \end{aligned}$$

6.3 Sufficiency

Assume that f and u are concave in k, c and $\lambda \geq 0$. This implies that the Hamiltonian is concave in k, c. Then, Pontryagin's necessary conditions (20) and (21) and (22), or (23) or (24) are sufficient.

6.4 Infinite horizon

Consider the infinite horizon problem

$$\begin{aligned} &\max_{c(t)_0^\infty} \int_0^\infty e^{-rt} u(k, c, t)\, dt && (55) \\ s.t.\, \dot{k} &= f(k, c, t) && (56) \\ k(0) &= \underline{k}, && (57) \end{aligned}$$

Pontryagin's conditions (20) and (21) are necessary also in the infinite horizon case, provided, of course, that there is a well defined solution. If there is a binding restriction on the state variable of the type $\lim_{T\to\infty} k(T) = \bar{k}$, this can help us pin down the solution. The finite horizon transversality conditions can, however, not immediately be used in the infinite horizon case. Suppose the *maximized* Hamiltonian is concave in k for every t, then the conditions (20) and (21) plus the infinite horizon transversality conditions

$$\begin{aligned}\lim_{T\to\infty} e^{-rT}\lambda(T)\,k(T) &= 0, \text{ and} &(58)\\ \lim_{T\to\infty} e^{-rT}\lambda(T) &\geq 0, &(59)\end{aligned}$$

provide a *sufficient* set of conditions for optimality. Often, the Hamiltonian is concave in k, c together. This is sufficient for the maximized Hamiltonian to be concave in k.

Sometimes a so called *No-Ponzi* condition helps us to make sure that the transversality conditions are satisfied. Suppose, for example, the pay-off $u(k,c,t) = u(c)$, that k represents debt of the agent and for simplicity that $f(k,c,t) = c + \rho k - w$, so debt increases by the difference between consumption plus interest payments ρk and the wage w. It is reasonable to assume that creditors demand to be repaid in a present value sense – the discounted value of future repayment should always be at least as large as debt. This is the No-Ponzi condition. When in addition, the agent prefers to pay back no more than he ows, the implication is.

$$\lim_{T\to\infty} e^{-\rho T}k(T) = 0. \tag{60}$$

To see this, solve

$$\dot{k}(t) - \rho k(t) = c(t) - w(t) \tag{61}$$

giving

$$\begin{aligned}e^{-\rho t}\left(\dot{k}(t) - \rho k(t)\right) &= e^{-\rho t}\left(c(t) - w(t)\right) &(62)\\ e^{-\rho t}k(t) &= \int_0^t \left(e^{-\rho s}\left(c(t) - w(t)\right)\right)ds + k(0) &(63)\\ \lim_{T\to\infty} e^{-\rho T}k(T) &= \int_0^\infty \left(e^{-\rho s}\left(c(t) - w(t)\right)\right)ds + k(0) &(64)\end{aligned}$$

So, the No-Ponzi requirement is that if the PDV of "mortgage" repayments is no smaller than initial debt, i.e., $-\int_0^\infty \left(e^{-\rho s}\left(c(t) - w(t)\right)\right)ds\ k(0) \geq k(0)$, then $\lim_{T\to\infty} e^{-\rho T}k(T) \leq 0$. Clearly, when marginal utility is strictly positive, the individual would never want to satisfy this with inequality, since he could then increase consumption. Therefore, $\lim_{T\to\infty} e^{-\rho T}k(T) = 0$.

The second necessary condition in (21) is now

$$\begin{aligned}-(\rho - r)\lambda(t) &= \dot{\lambda}(t) &(65)\\ \lambda(t) &= \lambda(0)\,e^{-(\rho-r)t}. &(66)\end{aligned}$$

So provided marginal utility is positive at $t = 0$, (59) is satisfied. Furthermore,

$$\begin{aligned}\lim_{T\to\infty} e^{-rT}\lambda(T)\,k(T) &= \lambda(0)\lim_{T\to\infty} e^{-rT}e^{-(\rho-r)T}k(T) && (67)\\ &= \lambda(0)\lim_{T\to\infty} e^{-\rho T}k(T) && (68)\\ &= 0. && (69)\end{aligned}$$

where the last equality is the No-Ponzi condition.

Sometimes, the sufficient conditions allow us to identify the optimal control as the stable manifold (saddle-path) leading to a saddle-point stable steady state. Consider again the problem

$$\max_{c(t)_0^\infty} \int_0^\infty e^{-rt}u(c)\,dt \qquad (70)$$

$$\text{s.t. } \dot{k} = f(k) - c,\ \ k(0) = \underline{k}, \qquad (71)$$

which we graphically analyzed in section 3.5 showing the existence of saddle-path and a steady state with

$$\begin{aligned} f'(k^{ss}) &= r && (72)\\ c^{ss} &= f(k^{ss}) && (73)\end{aligned}$$

Restating the current value Hamiltonian

$$\mathcal{H}(k,c,\lambda,t) = u(c) + \lambda(t)\left(f(k) - c\right), \qquad (74)$$

we note that if both $u(c)$ and $f(k)$ are concave, and $\lambda(t) \geq 0$, $\mathcal{H}(k,c,\lambda,t)$ is concave in k, c so the conditions for using the sufficiency result are satisfied. In addition to (41), (42), and $k(0) = \underline{k}$, we thus only need to verify that (58) and (59) are satisfied. This is straightforward,

$$\begin{aligned}\lim_{T\to\infty} e^{-rT}\lambda(T) &= \lim_{T\to\infty} e^{-rT}u'(c^{ss}) = 0, && (75)\\ \lim_{T\to\infty} e^{-rT}\lambda(T)\,k(T) &= \lim_{T\to\infty} e^{-rT}u'(c^{ss})\,k^{ss} = 0. && (76)\end{aligned}$$

6.5 Present value Hamiltonian

Sometimes, it is convenient to define the present value Hamiltonian, i.e., expressing everything in values as seen from time 0. In problem (1), the present value Hamiltonian is given by

$$\mathbf{H}(k,c,\mu,t) = e^{-rt}u(k,c,t) + \mu f(k,c,t), \qquad (77)$$

where $\mu(t)$ is the present shadow value of the state variable. In this case, the necessary conditions for optimality are

$$\begin{aligned} c^*(t) &= \arg\max_c \mathbf{H}(k, c, \mu, t) && (78) \\ -\dot{\mu}(t) &= \mathbf{H}_k(k, c, \mu, t) && (79) \\ \dot{k} &= \mathbf{H}_\mu(k, c, \mu, t) && (80) \end{aligned}$$

In the finite horizon case, the transversality conditions are the same in terms of $\mu(T)$ and $\lambda(T)$ .In the infinite horizon case, we note that

$$\mu(T) = e^{-rT}\lambda(T), \tag{81}$$

so the conditions (58) and (59) become

$$\begin{aligned} \lim_{T\to\infty} \mu(T) k(T) &= 0, \text{ and} && (82) \\ \lim_{T\to\infty} \mu(T) &\geq 0. && (83) \end{aligned}$$

6.6 Many state variables and controls

Having several state variables and controls pose no principle problem. Neither does pointwise discontinuities in the control variable. To generalize, suppose we have n state variables and n controls. An optimal control maximizes the Hamiltonian over all available controls $\mathbf{c}$

$$\mathbf{c}^*(t) = \arg\max_{\mathbf{c}} \mathcal{H}(\mathbf{k}, \mathbf{c}, \boldsymbol{\lambda}, t) \equiv u(\mathbf{k}, \mathbf{c}, t) + \sum_{i=1}^{n} \lambda_i f_i(\mathbf{k}, \mathbf{c}, t). \tag{84}$$

where λ_i is the shadow value associated with the state variable k_i. Each $\lambda_i(t)$, is continuous and satisfies the differential equation

$$r\lambda_i(t) - \dot{\lambda}_i(t) = \frac{\partial}{\partial k_i}\mathcal{H}(\mathbf{k}, \mathbf{c}, \boldsymbol{\lambda}, t), \tag{85}$$

except when c is discontinuous. For the transversality conditions, we have

$$\begin{aligned} & k_i(T) = \bar{k}_i \text{ case (1), or} && (86) \\ & \lambda_i(T) = 0 \text{ case (2), or} && (87) \\ & \lambda_i(T) \geq 0, \text{ and } \lambda_i(T)\left(k_i(T) - \bar{k}_i\right) = 0 \text{ case (3),} && (88) \end{aligned}$$

for end-conditions for state variable i belonging to case 1, 2 or 3.