The Growth and Stability Pact – the monster from Brussels

ECB is considering a fiscal policy rule to make sure that debt and deficits remain stable. The rule is of the form,

$$\dot{s} = \delta d - \sigma s,$$

where s is the primary surplus as share of GDP and d is the debt as share of GDP. δ measures how the fast consolidations (increasing the surplus) reacts to the debt and σ determines how fast a surplus (or deficit) returns to zero.

The dynamic budget constraint for the governments is

$$\begin{aligned} \dot{D} &= rD - S \\ \frac{\dot{D}}{D} &= r - \frac{S}{D} \end{aligned}$$

where r is the interest rate, D is debt and S the surplus.

Now,

$$\begin{aligned} \frac{\dot{d}}{d} &= \frac{\dot{D}}{D} - \frac{\dot{Y}}{Y} \\ &= r - \frac{S}{D} - g \\ &= r - \frac{s}{d} - g \end{aligned}$$

or

$$\dot{d} = d\,(r - g) - s \qquad (1)$$

We now want a differential equation in d only. For this purpose, we take the time derivative of the previous equation

$$\begin{aligned} \ddot{d} &= \dot{d}\,(r - g) - \dot{s} \\ &= \dot{d}\,(r - g) - (\delta d - \sigma s) \\ &= \dot{d}\,(r - g) - \delta d + \sigma \left(d\,(r - g) - \dot{d} \right) \\ &= \dot{d}\,(r - g - \sigma) - d\,(\delta - \sigma\,(r - g)) \end{aligned}$$

Implying

$$\ddot{d} - \dot{d}\,(r - g - \sigma) + d\,(\delta - \sigma\,(r - g)) = 0,$$

i.e., the debt share follows a homogeneous second order linear difference equation. The solution is therefore

$$d\,(t) = c_1 e^{\rho_1 t} + c_2 e^{\rho_2 t}, \qquad (2)$$

where the roots of system are

$$\begin{aligned} \rho_1 &= \frac{r - g - \sigma}{2} - \frac{1}{2}\sqrt{(r - g - \sigma)^2 - 4\,(\delta - \sigma\,(r - g))} \\ \rho_2 &= \frac{r - g - \sigma}{2} + \frac{1}{2}\sqrt{(r - g - \sigma)^2 - 4\,(\delta - \sigma\,(r - g))} \end{aligned}$$

When

$$\delta < \frac{(r-g-\sigma)^2}{4} + \sigma(r-g)$$

the roots are real. Furthermore, the largest root (i.e., ρ_2) falls in δ and is negative if

$$\delta > \sigma(r-g)$$

The ECB wanted global stability and no oscillations, requiring

$$\delta \in \left(\sigma(r-g), \frac{(r-g-\sigma)^2}{4} + \sigma(r-g)\right).$$

Suppose $\sigma > r-g$ but $\delta < \sigma(r-g)$. Then one root is negative and one is positive. We can still have convergence, but then the ECB must put some restrictions on the initial surplus so that (2) has $c_2 = 0$. To find this restriction, we note that

$$\dot{d}(t) = c_1\rho_1 e^{\rho_1 t} + c_2\rho_2 e^{\rho_2 t}. \tag{3}$$

Given $d(0)$ and $c_2 = 0$, we have two unknowns (c_1 and $\dot{d}(0)$) Fortunately we have two equations, (2) and (3) evaluated at zero.

$$\begin{aligned} d(0) &= c_1 \\ \dot{d}(0) &= c_1\rho_1 \\ &= d(0)\rho_1 \end{aligned}$$

So if, $\dot{d}(0) = d(0)\rho_1$, the deficit share converges to zero. We may want to express this in terms of the primary surplus by using (1)

$$\begin{aligned} s(0) &= d(0)(r-g-\rho_1) \\ s(0) &= d(0)\left(\frac{r-g+\sigma}{2} - \frac{1}{2}\sqrt{(r-g-\sigma)^2 - 4(\delta-\sigma(r-g))}\right). \end{aligned}$$

If instead $\delta > \frac{(r-g-\sigma)^2}{4} + \sigma(r-g)$, the system oscillates but if the real part of the roots, $\frac{r-g-\sigma}{2} < 0$, the system converges. The solution when roots are complex is

$$d(t) = e^{at}(c_1\cos(bt) + c_2\sin(bt)),$$

where

$$\begin{aligned} a &= \frac{r-g-\sigma}{2}, \\ b &= \frac{1}{2}\sqrt{4(\delta-\sigma(r-g)) - (r-g-\sigma)^2}. \end{aligned}$$

As we see, the speed in the oscillations increase in δ, i.e., in the parameter that determines how fast the surplus reacts to the debt.

Again, we can find the integration constants c_1 and c_2 from knowledge of $d(0)$ and $\dot{d}(0)$.

$$d(0) = (c_1\cos(0) + c_2\sin(0)) = c_1,$$

and

$$\begin{aligned}
\dot{d}\left(t\right) &= e^{at}\left(a\left(c_1\cos bt + c_2\sin bt\right) + b\left(c_2\left(\cos bt\right) - c_1\left(\sin bt\right)\right)\right) \\
\dot{d}\left(0\right) &= \left(a\left(c_1\cos 0 + c_2\sin 0\right) + b\left(c_2\left(\cos 0\right) - c_1\left(\sin 0\right)\right)\right) \\
&= ac_1 + bc_2 \\
&= ad\left(0\right) + bc_2 \\
&\rightarrow \quad c_2 = \frac{\dot{d}\left(0\right) - ad\left(0\right)}{b}
\end{aligned}$$

implying

$$\begin{aligned}
d\left(t\right) &= e^{at}\left(d\left(0\right)\cos\left(bt\right) + \frac{\dot{d}\left(0\right) - ad\left(0\right)}{b}\sin\left(bt\right)\right), \\
&= e^{at}\left(d\left(0\right)\cos\left(bt\right) + \frac{d\left(0\right)\left(r-g-a\right) - s\left(0\right)}{b}\sin\left(bt\right)\right),
\end{aligned}$$

where we used the budget constraint $\dot{d} = d\left(r-g\right) - s$ evaluated at 0 to substitute for $\dot{d}\left(0\right)$.