8 Note on constant hazard models

Consider as an example a pool of unemployed, with measure (size) at time t given by $x(t)$. Suppose also that there is an outflow of people from the unemployment pool (we are for now disregarding the inflow of new unemployed). The outflow is determined by an assumption that there is a constant probability per unit of time, denoted h to get hired. Using this we can derive a law-of-motion for $x(t)$. Over a small (infinitesimal) interval of time dt, we have

$$\begin{aligned} x(t+dt) &= x(t)(1-hdt) \quad (1) \\ x(t+dt)-x(t) &= -x(t)\,hdt \quad (2) \\ \lim_{dt\to 0}\frac{x(t+dt)-x(t)}{dt} &\equiv \dot{x}(t) = -hx(t). \quad (3) \end{aligned}$$

This is a simple differential equation with a solution

$$x(t) = x(0)\,e^{-ht}. \quad (4)$$

Now, consider an individual who is unemployed at time 0 and the random variable s denote the time she will stay unemployed. Let us now derive the probability density function $f(s)$ and let $F(s)$ denote the cumulative distribution function, i.e., $F(s)$ is the probability the unemployment spell is no longer than s. Clearly,

$$F(s) = \int_0^s f(t)\,dt. \quad (5)$$

From (4) we know that at time s, a share

$$\frac{x(t)}{x(0)} = e^{-hs}, \quad (6)$$

remains unemployed and since hiring is completely random,

$$1-\frac{x(t)}{x(0)} = 1-e^{-hs} = F(s), \quad (7)$$

and consequently

$$f(s) = he^{-hs}. \quad (8)$$

Let us also verify that the probability of finding a job per unit of time, conditional on not having found it stays constant. This probability is

$$\frac{f(t)}{1-F(t)} = \frac{he^{-ht}}{e^{-ht}} = h. \quad (9)$$

We can now compute the average spell length as

$$\int_0^\infty sf(s)\,ds = \int_0^\infty she^{-hs}ds. \tag{10}$$

Using the formula for integration by parts,

$$\begin{aligned} \int_0^\infty she^{-hs}ds &= \left[-se^{-hs}\right]_0^\infty - \int_0^\infty -e^{-hs}ds && (11) \\ &= 0 - 0 - \left[\frac{e^{-hs}}{h}\right]_0^\infty = \frac{1}{h}. && (12) \end{aligned}$$

Similarly, we can compute the median length m, i.e., solving $F(m) = 1/2$ from

$$\begin{aligned} 1/2 &= 1 - e^{-hm} && (13) \\ &\rightarrow e^{-hm} = 1/2 && (14) \\ -hm &= -\ln 2 && (15) \\ m &= \frac{\ln 2}{h} \approx \frac{.69}{h} && (16) \end{aligned}$$

This is sometimes called the *rule of 69;* expressing h in percent per unit of time. The half-life is found by dividing 69 by h. For example, if the probability of finding job is 5% per week. It takes $69/5 \approx 14$ weeks before half the pool of unemployed have found jobs and the average unemployment spell is $1/0.05 = 20$ weeks.

Another example; it is often found that differences in GDP between countries (after controlling for differences is savings and schooling) is closing by 3% per year. Then, half the difference is left after $69/3 = 23$. The rule of 69 also works when something is growing. If a bank account yields 4% return per year, it takes $69/4 \approx 17$ years for it to double.