A note on discounting.

Suppose utility is additively separable over time and the future is discounted. Then, in the discrete time case, the objective function of an individual can be written

$$\sum_t R_t U(c_t)\,, \tag{1.1}$$

where R_t is a decreasing sequence of discount factors. In the special case of geometric discounting, R_t follows a simple difference equation,

$$R_{t+1} = \beta R_t \tag{1.2}$$

where we refer to β as the (per period) (subjecive) discount *factor*, where the words "per period" and "subjective" often are omitted. Alternatively we can write

$$R_{t+1} = \frac{1}{1+r} R_t \tag{1.3}$$

where r is called the (per period) (subjective) discount *rate*. In both cases, the difference equation is easy to solve

$$\begin{aligned} R_t &= C\beta^t, \text{ or} \\ R_t &= C(1+r)^{-t} \end{aligned} \tag{1.4}$$

where the constant C is unity, since we want $R_0 = 1$.

Defining $r/(1+r) \equiv \rho$, we can rewrite (1.3) as

$$R_{t+1} - R_t \equiv \Delta R_{t+1} = \left(\frac{1}{1+r} - 1\right) R_t = -\rho R_t\,. \tag{1.5}$$

Now, consider the continuous time case, where the objective function is

$$\int R_t U(c_t)\,dt\,. \tag{1.6}$$

The continous time analogue of (1.5) is

$$\dot{R}_t = -\rho R_t \tag{1.7}$$

with the solution

$$R_t = e^{-\rho t}\,, \tag{1.8}$$

where, again, the integration constant is unity. Both (1.4) and (1.8) has the very important property that the ratio between the discount factors at two different points in time (t and s) depends only on the difference in time between the two points;

$$\frac{R_t}{R_s} = \frac{\beta^t}{\beta^s} = \beta^{t-s}, \tag{1.9}$$

or

$$\frac{R_t}{R_s} = \frac{e^{-\rho t}}{e^{-\rho s}} = e^{-\rho(t-s)}. \tag{1.10}$$

This implies that the trade of between consuming at two different future dates stays constant as time passes and the two future dates come closer. This is needed to have dynamic consistency, i.e., that the agent does not change his mind regarding his (consumption) plans.

We can also derive (1.8) formally in the following way; Consider a discrete time setup with the time interval given by Δt. Suppose the discount rate in discrete time is ρ per unit of time (for example 12% per year). Over a time interval Δt, the discount rate is $\rho\Delta t$ so if, for example, Δt equals a month, the discount rate over a time interval is 1%. Then, the discount factor applying at some date t, is

$$\left(\frac{1}{1+\rho\Delta t}\right)^{t/\Delta t}. \tag{1.11}$$

Now, let the length of each time interval go to zero. Calculating this limit, we get

$$\lim_{\Delta t \to 0}\left(\frac{1}{1+\rho\Delta t}\right)^{t/\Delta t} = e^{-\rho t}. \tag{1.12}$$