1. **A dynamic programming problem**
Consider the following problem:

$$\begin{array}{ll} \max_{\{c_t\}} & \sum_{t=0}^{\infty} \beta^t U(c_t) \\ s.t. & A_{t+1} = RA_t - c_t, \\ & A_0 = A, \\ & A_t, c_t \geq 0, \forall t, \end{array}$$

with parameters satisfying $0 < R\beta^2 < 1$, $R > 1$, $0 < \beta < 1$ and an associated value function $V(A_t)$.

(a) Substitute from the dynamic budget constraint to get rid of c_t. Write the Bellman equation as a maximum over A_{t+1} and the first order condition for a maximum.

(b) Let U be the CRRA function $U(c_t) = 2\sqrt{c_t}$ and guess that $V(A_t) = 2\gamma\sqrt{A_t}$ solves the Bellman equation for some constant γ. Use this to show that along the optimal path

$$A_{t+1} = \frac{(\beta\gamma)^2}{1 + (\beta\gamma)^2} RA_t$$

(c) Verify that the guess was correct and show that

$$\gamma = \sqrt{\frac{R}{1 - R\beta^2}}.$$

(d) Show that the last constraint $A_t \geq 0 \forall t$ is satisfied and solve for the path of c_t.

(e) Can you prove that you have found the solution to the problem? (Hint: If $V(x)$ solves the Bellman equation *and* $\lim_{T\to\infty} \beta^T V(x_T) = 0$ *for all* permissible paths of x, $V(x)$ is the correct value function.)

2. Safe and risk assets
Consider the following problem:

$$\max_{c_t, \omega_t} E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \tag{1}$$

subject to the constraints

$$\begin{aligned} A_{t+1} &= (A_t - c_t)\left((1+r)\,\omega_t + (1 + z_{t+1})\,(1 - \omega_t)\right) && (2) \\ A_0 &= A > 0. && (3) \end{aligned}$$

where c_t is consumption in period t, A_t is total assets in the beginning of period t, ω_t is the share of assets that is invested in a riskless asset yielding a constant return r, and z_{t+1} is the stochastic yield on a risky asset. z_{t+1} is identically and independently distributed (i.i.d.) over time. σ is the coefficient of relative risk aversion and is strictly positive and finite.

1. (a) Write a the Bellman equation for the problem with its associated first order conditions.
   (b) Use the Envelope condition to derive the Euler equation.
   (c) Is ω_t going to be constant over time, why/why not?
   (d) Guess that the value function has the form

$$V(A_t) = k\frac{A_t^{1-\sigma}}{1-\sigma}. \tag{4}$$

   for some, still unknown, value k. Find the consumption function without solving for k.
   (e) Verify the guess in (4) was correct and find an expression for k in terms of parameters. Use the following notation to simplify your expressions;

$$\theta \equiv E_t\left((1+r)\,\omega_t^* + (1+z_{t+1})\,(1-\omega_t^*)\right) \tag{5}$$

   where ω_t^* is the optimal value for ω.

3. *Simplified RBC Model*

In this problem set you are going to construct a small real business cycle model. Let there be representative consumer with log utility who owns a firm with Cobb-Douglas technology. At time t the problem for the consumer is then

1.

$$\max E_t \sum_{s=0}^{\infty} \beta U\left(C_{s+t}\right) \tag{6}$$

subject to the constraints

$$C_t + K_{t+1} = Y_t \tag{7}$$

$$Y_t = Z_t K_t^{1-\alpha} \tag{8}$$

where C is consumption, K is the capital stock, Y is output, Z is a stochastic productivity shock and $\alpha \in (0,1)$.

   (a) Write down the Bellman equation. Make sure you use the correct state variable. Hint, is K_t a sufficient statistic to calculate the expected PDV of utility?
   (b) Assume utility can be represented by the log function ($U(C) = \ln(C)$). Suppose that this, and and iid assumption on Z implies that the value function can be written $V(X) = k_0 ln(X) + k_1$.where you replace X by the true state variable. Find the consumption rule, verify that the guess is correct and solve for k_0.
   (c) Let lower case letters denote logaritms. Assume that z_t is i.i.d. over time. Find the stochastic process for c. What is the autocorrelation coefficient of output growth?
   (d) What is the correlation between consumption and output?