Formulas for MathII Exam.

These sheets will be distributed with the exam. Do not bring your own.

1. Complex numbers and trigonometric functions

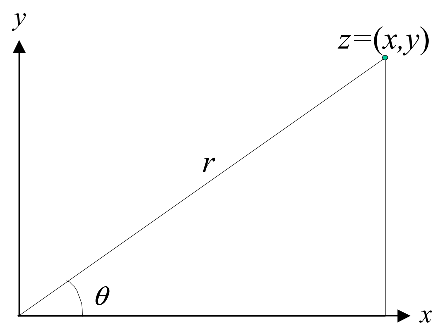
Measured in radians we have $\sin(0) = 0$, $\cos(0) = 1$,$\sin(\pi/2) = 1$, $\cos(\pi/2) = 0$, $\sin(\pi) = 0$, $\cos(\pi) = -1$, $\sin\left(\frac{3\pi}{2}\right) = -1$, $\cos\left(\frac{3\pi}{2}\right) = 0$, $\frac{d\sin(x)}{dx} = \cos(x)$ and $\frac{d\cos(x)}{dx} = -\sin(x)$.

The formula for the solution to a quadratic equation $ax^2 + bx + c = 0$, is

$$x = -\frac{b}{2a} \pm \frac{\sqrt{(b^2 - 4ac)}}{2a}.$$

In the right-angled triangle in the figure, we have

$$\cos(\theta) = \frac{x}{r}, \sin(\theta) = \frac{y}{r}, r = \sqrt{x^2 + y^2}.$$



Let z be the complex number (x, y) , then z satisfies $z = re^{i\theta}$ where r is the modulus of z and θ is the argument.

2. Differential equations.

For a linear differential equation with one of the roots given by r_n, the associated homogeneous solution is

$$y(t) = e^{r_n t}.$$

For a linear differential equation with a complex pair of roots $r_{1,2} = a \pm bi$, the associated homogeneous solution is

$$y(t) = e^{at}\left(\bar{c}_1 \cos(bt) + \bar{c}_2 \sin(bt)\right).$$

For a linear differential equation with repeated roots $r_1, ...r_k$ all equal to r, the associated homogeneous solution is

$$y(t) = c_1 e^{rt} + c_2 t e^{rt} + ... + c_k t^{k-1} e^{rt}.$$

For the homogeneous linear system of differential equations,

$$\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} = \mathbf{A} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix},$$

where A is assumed to have distinct eigenvalues different from zero, the solution is

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \mathbf{B}^{-1} \begin{bmatrix} e^{r_1 t} & 0 \\ 0 & e^{r_2 t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

where $\mathbf{B}^{-1}$ is the matrix of eigenvectors of $\mathbf{A}$ and r_1 and r_2 are the eigenvalues of $\mathbf{A}$.

3. Difference equations

For a linear differential equation with one of the roots given by r_n, the associated homogeneous solution is

$$x_t = c_1 r_1^t.$$

For a linear differential equation with a pair of complex roots given by $r_{1,2} = a \pm bi$, the associated homogeneous solution is

$$x_t = |r|^t \left(\tilde{c}_1 \cos t\theta + \tilde{c}_2 \sin t\theta\right)$$

where$|r| = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1} \frac{b}{a}$.

For a linear differential equation with repeated roots $r_1, ...r_k$, all equal to r, the associated homogeneous solution is

$$x_t = r^t \left(c_1 + tc_2 + ... + t^{k-1} c_k\right).$$

For the homogeneous linear system of differential equations,

$$\begin{bmatrix} y_{1,t+1} \\ y_{2,t+1} \end{bmatrix} = \mathbf{A} \begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix},$$

where A is assumed to have distinct eigenvalues different from zero, the solution is

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \mathbf{B}^{-1} \begin{bmatrix} r_1^t & 0 \\ 0 & r_2^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

4. Linear algebra

The eigenvalues of a matrix

$$\mathbf{A} \equiv \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

are

$$r_{1,2} = \frac{a_{11} + a_{22} \pm \sqrt{\left(a_{11} - a_{22}\right)^2 + 4a_{12}a_{21}}}{2}.$$

If r_i is an eigenvalue of $\mathbf{A}$, the associated eigenvector $\mathbf{e}_i$ satisfies

$$\mathbf{A}\mathbf{e}_i = r_i \mathbf{e}_i.$$

The inverse of the matrix is given by

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{-1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} -a_{22} & a_{12} \\ a_{21} & -a_{11} \end{bmatrix}.$$