1 Linear Hamiltonians

Consider the following problem

$$\max_{c(t)_0^T \in [0,1]} \int_0^T e^{-rt} c dt$$

$$\begin{aligned} \text{s.t } \dot{k} &= Ak - c \\ k(0) &= k_0 > 0. \\ k(T) &\geq 0 \end{aligned}$$

Then the Hamiltonian is

$$\mathcal{H}(k, c, \lambda, t) = c + \lambda (Ak - c). \tag{1}$$

Clearly, the Hamiltonian is linear in c, so

$$c^* = \begin{cases} 0 \text{ if } \lambda > 1 \\ 1 \text{ if } \lambda \leq 1 \end{cases}$$

maximizes the Hamiltonian.

Now, the second decessary condition is

$$\lambda A = r\lambda - \dot{\lambda}$$

or

$$\frac{d\lambda}{dt} = \lambda (r - A).$$

implying

$$\lambda(t) = C_1 e^{(r-A)t}.$$

Clearly, $\lambda(t) > 0$ for all t, unless $\lambda(0) = 0$, implying that the transversality constraint

$$\lambda(T) k(T) = 0$$

implies

$$k(T) = 0,$$

if $\lambda(0) > 0$. Note that if $\lambda(0) = 0$, it is optimal to set $c(t) = 1$ for all t.

Now, consider the case when $A > r$. Then, $\lambda(t)$ is falling, and we tentatively guess that there is a $t_1$such that

$$\lambda(t_1) = 1.$$

Then,

$$\dot{k} = \begin{cases} Ak \text{ if } t < t_1 \\ Ak - 1 \text{ if } t > t_1 \end{cases}$$

implying

$$k(t) = \begin{cases} k(0)\, e^{At} \text{ if } t < t_1 \\ \frac{1}{A}\left(1 - e^{A(t-t_1)}\right) + e^{At} k_0 \text{if } t > t_1 \end{cases}$$

We now impose $k(T) = 0$.

$$\begin{aligned} \frac{1}{A}\left(1 - e^{A(T-t_1)}\right) + e^{AT}(k_0) &= 0 \\ \rightarrow \quad t_1 = T - \frac{\ln\left(1 + e^{AT} k_0 A\right)}{A} \end{aligned}$$

If this is negative, our guess was wrong and $\lambda(t) < 1$ for all t so $c(t) = 1$ for all t. Note that in that case, we could have $\lambda(0) = 0$, which would occur if $\frac{1}{A}\left(1 - e^{At}\right) + e^{At}(k_0) > 0$, i.e., one cannot eat all capital during the period $[0, T]$.

Finally, is it possible that $\lambda(t) > 1$ for all t? In that case, we know $k(T)$ must be zero, but thus is inconsistent with $\lambda(t) > 1$ and thus $c(t) = 0$ for all t.

The analysis is similar in the case when $r > A$.The only difference is that $\lambda(t)$ is increasing, so consumption takes place before some threshold time.