T-mappings

Instead of using time as an argument of the value function, let's use time subscripts. We can then write the Bellman equation as

$$V_t(k_t) = \begin{cases} \max_{c_t}\left(u(k_t, c_t) + \beta V_{t+1}(k_{t+1})\right) \\ s.t. k_{t+1} = f(k_t, c_t) \end{cases}$$

Now, let us define an operator that maps next period value functions, $V_{t+1} : K \to R$, (where K is the state space, i.e., the set of possible values for the state variable), into functions that provides the current value associated with all k_t in the state space. Thus, the operator, which we will call, T, maps elements of the space of value functions, call that space C, back into the same space, i.e., $T : C \to C$.[1]

Formally, we define the T mapping as

$$TV_{t+1} : C \to C \equiv \begin{cases} \max_{c_t}\left(u(k_t, c_t) + \beta V_{t+1}(k_{t+1})\right) \\ s.t. k_{t+1} = f(k_t, c_t) \end{cases}.$$

When we want to indicate that the mapped function, TV_{t+1}, is a function of, e.g., k_t,we append (k_t);

$$TV_{t+1} = TV_{t+1}(k_t).$$

Note that while V_{t+1} is a function of k_{t+1}, TV_{t+1} is a function of k_t.

Let us take an example. Suppose $u(k_t, c_t) = \ln c_t$ and $f(k_t, c_t) = k_t - c_t$ $\forall t$. Let's us now see what the T operator does. Take a particular element in C, for example $\ln k : R^+ \to R$. So here the state space $K = R^+$. Now,

$$T \ln k_t = \begin{cases} \max_{c_t}\left(\ln c_t + \beta(\ln k_{t+1})\right) \\ s.t. k_{t+1} = k_t - c_t \end{cases}$$

The first order condition is

$$\frac{1}{c_t} = \frac{\beta}{k_t - c_t} \to c_t = \frac{k_t}{1+\beta}$$

thus,

$$\begin{aligned} T \ln k_t &= \ln \frac{k_t}{1+\beta} + \beta\left(\ln\left(k_t - \frac{k_t}{1+\beta}\right)\right) \\ &= \left((1+\beta)\ln k_t + \beta \ln \beta - (1+\beta)\ln(1+\beta)\right) \end{aligned}$$

which is another function $R^+ \to R$. So, we see that the T maps functions into (potentially) other functions, concluding the example.

Using the T operator, the Bellman equation in period t can be written

$$V_t = TV_{t+1},$$

[1] Later on, we will make assumptions such that we can further restrict the space of possible valute functions.

or, equivalently,

$$V_t(k_t) = TV_{t+1}(k_t).$$

Furthemore, in the next period, the next period's Bellman equation is

$$V_{t+1} = TV_{t+2}.$$

Thus,

$$V_t = TV_{t+1} = T^2 V_{t+2}.$$

The meaning of $T^2 V_{t+2}$ in words is; give me (I am T^2) a value function that applies in period $t+2$ (you don't need to say anything about what k_{t+2} is going to be), and I give you a function that tells you the value in period t associated with all values of k_t in the state space. Formally;

$$\begin{aligned} T^2 V_{t+2} &= \left\{ \begin{array}{c} \max_{c_t} \left(u\left(k_t, c_t\right) + \beta\left(TV_{t+2}\left(k_{t+1}\right)\right)\right) \\ s.t. k_{t+1} = f\left(k_t, c_t\right) \end{array} \right. \\ &= \left\{ \begin{array}{c} \max_{c_t} \left(u\left(k_t, c_t\right) + \beta\left(\max_{c_{t+1}}\left(u\left(k_{t+1}, c_{t+1}\right) + \beta V_{t+2}\left(k_{t+2}\right)\right)\right)\right) \\ s.t. k_{t+1} = f\left(k_t, c_t\right), k_{t+2} = f\left(k_{t+1}, c_{t+1}\right) \end{array} \right. . \end{aligned}$$

Now, suppose in the time autonomuous case that we can find a limiting function

$$V(k_t) \equiv \lim_{s \to \infty} T^s V_{t+s}(k_t)$$

then, as shown in the lecture notes, this function satisfies the Bellman equation

$$V(k_t) = TV(k_t)$$

i.e., it is a fixed point of the T operator. This means that in the the space C, V is an elements such that T maps back onto the same element. If T is a contraction mapping on C, this element exists and is unique.