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Systems of differential equations

1. Systems of linear equations with constant coefficients

Consider a system of two linear equations with constant coefficients:

$$x_1'(t) = a_{11}x_1(t) + a_{12}x_2(t) + b_1$$
$$x_2'(x) = a_{21}x_1(t) + a_{22}x_2(t) + b_2$$

This system may be written more compactly using matrix notation:

$$\underline{x}'(t) = Ax(t) + \underline{b} \tag{1.1}$$

where

$$\underline{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{12} \end{pmatrix} \quad \text{and} \quad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

We assume that the determinant $\Delta \overset{\text{def}}{=} \det(A) \neq 0$. It is then easy to see that a *steady state* solution—i.e., a solution independent of t—to (1.1) is $\underline{x}^* = -A^{-1}\underline{b}$.

For any $\underline{x}(t)$ we can define $\underline{z}(t)$ by $x(t) = \underline{z}(t) + \underline{x}^*$; substituting into (1.1) yields

$$\underline{z}'(t) = A(\underline{z}(t) + \underline{x}^*) + \underline{b} = A\underline{z}(t) - A(A^{-1}\underline{b}) + \underline{b} = A\underline{z}(t)$$

i.e., $\underline{x}(t)$ is a solution to (1.1) if and only if $\underline{x}(t) = \underline{z}(t) + \underline{x}^*$, where $\underline{x}^* = -A^{-1}\underline{b}$ and $\underline{z}(t)$ is a solution to the *homogeneous* equation

$$\underline{z}(t) = A\underline{z}(t) \tag{1.2}$$

We now proceed to find the general solution.

Theorem: *Let* $\Phi(t) = \begin{pmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{pmatrix}$ *be a solution to*

$$\Phi'(t) = A\Phi(t) \tag{1.3}$$

where

$$\det \Phi(t) = \begin{vmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{vmatrix} \neq 0 \quad \textit{for all } t.$$

Then the general solution to equation (1.2) is $\underline{z}(t) = \Phi(t)\underline{c}$, *where* $\underline{c}$ *is a constant* $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$. *In other words, the general solution to equation (1.1) is*

$$\underline{x}(t) = \underline{\phi}_1 c_1 + \underline{\phi}_2 c_2 - A^{-1}\underline{b}$$

where $\underline{\phi}_j$ *is the j:th column of* Φ *and* c_1, c_2 *are arbitrary constants.*

Proof: For any $\underline{z}(t)$, define $\underline{c}(t)$ by $\underline{z}(t) = \Phi(t)\underline{c}(t)$. This is possible, since $\det \Phi(t) \neq 0$. Substituting into (1.1) gives

$$\Phi'(t)\underline{c}(t) + \Phi(t)\underline{c}'(t) = A\Phi(t)\underline{c}(t)$$
$$\Longleftrightarrow$$
$$A\Phi(t)\underline{c}(t) + \Phi(t)\underline{c}'(t) = A\Phi(t)\underline{c}(t)$$
$$\Longleftrightarrow$$
$$\Phi(t)\underline{c}'(t) = 0 \quad \Longleftrightarrow \quad \underline{c}'(t) = 0 \quad \Longleftrightarrow \quad \underline{c}(t) = \underline{c} = \text{constant}$$

so we have $\underline{z}(t) = \Phi(t)\underline{c}$, *Q.E.D.*

2. Finding $\Phi(t)$

Note that $\Phi(t)$ is a solution to (1.3) if and only if each of the two columns $\underline{\phi}_1(t)$, $\underline{\phi}_2(t)$ are solutions to (1.1). Let us try to find a solution to (1.1) of the form $\underline{\phi}(t) = \underline{v}e^{rt}$ for some constant column matrix $\underline{v}$ and real number r. We get

$$\underline{v}\, re^{rt} = A\underline{v}e^{rt} \quad \Longleftrightarrow \quad A\underline{v} = r\underline{v}$$

i.e., we have found a solution iff. $\underline{v}$ is an eigenvector of A and r is the corresponding eigenvalue. The procedure is thus: if the *characteristic equation* $r^2 - \operatorname{Tr} A + \det A = 0$ has two distinct real roots r_1, r_2, and $\underline{v}_1$, $\underline{v}_2$ are the corresponding eigenvectors, then we have two solutions $\underline{\phi}_j(t) = \underline{v}_j e^{r_j t}$. Since eigenvectors corresponding to different eigenvalues are linearly independent, it follows that the determinant $\det(\underline{\phi}_1, \underline{\phi}_2) \neq 0$. Thus

Theorem: *If the characteristic equation* $r^2 - \operatorname{Tr} A + \det A = 0$ *has two distinct real roots* r_1, r_2, *and* $\underline{v}_1$, $\underline{v}_2$ *are the corresponding eigenvectors, then the general solution to equation (1.1) is (assuming* $\det A \neq 0$) :

$$\underline{x}(t) = c_1 e^{r_1 t}\, \underline{v}_1 + c_2 e^{r_2 t}\, \underline{v}_2 - A^{-1}\underline{b}$$

where c_1 *and* c_2 *are arbitrary constants.*

If the characteristic equation has a double root, or complex roots, we state the general solution to equation (1.1) without proof:

Theorem: *If the characteristic equation* $r^2 - \mathrm{Tr}\, A + \det A = 0$ *has a double root* r, *then the general solution of (1.1) is*

$$\underline{x}(t) = \underline{v}_1\, e^{r_1 t} + \underline{v}_2\, te^{r_2 t} - A^{-1}\underline{b}$$

where

$$\underline{v}_1 \text{ is an arbitrary column matrix and } \underline{v}_2 = (A - Ir)\underline{v}_1$$

If the characteristic equation has complex roots $\alpha \pm i\beta$, *then the general solution is*

$$\underline{x}(t) = \underline{v}_1\, e^{\alpha t}\cos\beta t + \underline{v}_2\, e^{\alpha t}\sin\beta t - A^{-1}\underline{b}$$

where

$$\underline{v}_1 \text{ is an arbitrary column matrix and } \underline{v}_2 = \frac{1}{\beta}(A - I\alpha)\underline{v}_1$$

3. Characterization of solutions.

The solutions can be depicted in a *phase diagram* where the curves $(x_1(t), x_2(t))$ are plotted in an x_1, x_2 diagram. The solution curves are called *trajectories.* We can distinguish between a number of cases. Let $T = \mathrm{Tr}\, A$ and $\Delta = \det A$. If $4\Delta > T^2$, then the characteristic equation features complex roots, and any trajectory is a *stable spiral* if $\alpha < 0$. In this case the trajectory spiral in towards the stationary solution $\underline{x}^*$. If instead $\alpha > 0$ we have *unstable* spirals; any trajectory will spiral out from $\underline{x}^*$. In either case, the spirals will turn in a positive—i.e., counter clockwise—direction if $a_{21} > 0$ and in negative—clockwise—direction if $a_{21} < 0$. If α happens to be exactly equal to zero, then the trajectories will go around the stationary solution $\underline{x}^*$ in ellipses; we have *periodic* solutions.

If the characteristic equation has distinct real roots, which is the case if $4\Delta < T^2$, then there are three cases depending on the location of the roots. Define $\psi(r) = r^2 - rT + \Delta$. This function is a parabola with it's minimum located at $r = T/2$.

(1.1) $\Delta > 0$ and $T < 0$. In this case $\psi(r)$ has it's minimum point $T/2$ to the left of $r = 0$ where $\psi(T/2) = -T^2/4 + \Delta < 0$ and $\psi(0) = \Delta > 0$. It follows that both roots r_1, r_2 are negative. The phase portrait is a *stable node.* All trajectories will converge towards the stationary solution $\underline{x}^*$ and will become tangent to the eigenvector corresponding to the eigenvalue closest to zero as $t \to \infty$.

(1.2) $\Delta > 0$ and $T > 0$. In this case $\psi(r)$ has it's minimum point $T/2$ to the right of $r = 0$ where $\psi(T/2) = -T^2/4 + \Delta < 0$ and $\psi(0) = \Delta > 0$. It follows that both roots r_1, r_2 are positive. The phase portrait is an *unstable node.* All trajectories will diverge away from the stationary solution $\underline{x}^*$ and will become tangent to the eigenvector corresponding to the eigenvalue closest to zero as $t \to -\infty$.

(1.3) $\Delta < 0$. In this case $\psi(0) = \Delta < 0$, it follows that one root is negative and the other positive; say $r_1 < 0$ and $r_2 > 0$. The phase portrait is a *saddle* *point.* Almost all trajectories will diverge away from the stationary solution $\underline{x}^*$; the only exception are the solutions $\underline{x}(t) = c_1 e^{r_1 t}\, \underline{v}_1 + c_2 e^{r_2 t}\, \underline{v}_2 - A^{-1}\underline{b}$ where $c_2 = 0$. These solutions $\underline{x}(t) = c_1 e^{r_1 t}\, \underline{v}_1 - A^{-1}\underline{b}$ are called *saddle paths* and play an important rule in many macro economic models.

Since $2\alpha = T$ we have in summary:

$T < 0,\ \ 4\Delta > T^2,\ \ a_{21} \begin{cases} > 0 \\ < 0 \end{cases}$	stable spiral $\begin{cases} \text{counter-clockwise} \\ \text{clockwise} \end{cases}$
$T > 0,\ \ 4\Delta > T^2,\ \ a_{21} \begin{cases} > 0 \\ < 0 \end{cases}$	unstable spiral $\begin{cases} \text{counter-clockwise} \\ \text{clockwise} \end{cases}$
$T > 0,\ \ 0 < 4\Delta < T^2$	unstable node
$T < 0,\ \ 0 < 4\Delta < T^2$	stable node
$\Delta < 0$	saddle point

4. Linearization and isoclines.

Consider a non-linear autonomous system of equations

$$\begin{aligned} x_1'(t) &= F(x_1, x_2) \\ x_1'(t) &= G(x_1, x_2) \end{aligned} \tag{4.1}$$

and let (x_1^*, x_2^*) be a stationary solution, i.e., a solution to $F(x_1^*, x_2^*) = G(x_1^*, x_2^*) = 0$. We can then *linearize* the equation:

$$\underline{x}(t) = A\underline{x}(t) - A\underline{x}^* \tag{4.2}$$

where

$$A = \begin{pmatrix} F_{x_1}(x_1^*, x_2^*) & F_{x_2}(x_1^*, x_2^*) \\ G_{x_1}(x_1^*, x_2^*) & G_{x_2}(x_1^*, x_2^*) \end{pmatrix}$$

Theorem: *In any of the cases listed in the summary in the previous section, the the solutions to the linearized system (4.2) are good approximations to the solutions to the nonlinear system (4.1) as long as they stay close to the equilibrium* (x_1^*, x_2^*).

Another help in studying nonlinear systems is the plotting of *isoclines.* The trajectories will traverse the curve $G(x_1, x_2)$—the zero isocline—horizontally, and the curve $F(x_1, x_2)$—the infinity isocline—vertically. These two isoclines divides the x_1, x_2 plane into regions, and in each of these the derivatives of $x_1(t)$ and $x_2(t)$ are constant. These observations are very helpful in the study of the qualitative behavior of the solutions of (4.1).