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Comments on difference equations

1. Backward and forward solutions to differences

Let A_t be an unknown sequence satisfying

$$A_t - A_{t-1} = q_t \tag{1.1}$$

where q_t is a known sequence. Replace t by k in (1.1) and add from 1 to t :

$$\sum_1^t (A_k - A_{k-1}) = \sum_1^t q_k$$

The first sum is "telescoping"—all A_k :s except the first and the last cancel—hence

$$A_t - A_0 = \sum_1^t q_k$$

i.e.,

$$A_t = A + \sum_1^t q_k \tag{1.2}$$

where $A = A_0$ is an arbitrary constant. This is true also for $t \leq 0$ with the summation convention

$$\sum_1^t q_k = \begin{cases} 0 & \text{if } t = 0 \\ -q_0 - q_{-1} - \cdots - q_{-t+1} & \text{if } t < 0 \end{cases}$$

In many applications in economics one prefers another representation of the solution. *If* the sum $\sum_{-\infty}^0 q_k$ converges, then we can replace A by the new constant $A + \sum_{-\infty}^0 q_k$ in (1.2) to get

$$A_t = A + \sum_{-\infty}^t q_k \tag{1.3}$$

The solution (1.3) is the *backward* solution to (1.1). In a similar manner, we can define the *forward* solution to (1.1) as

$$A_t = A - \sum_{t+1}^\infty q_k \tag{1.4}$$

which is well defined if the sum $\sum_0^\infty q_k$ converges; again A is an arbitrary constant. Indeed, replacing A by $A - \sum_0^\infty q_k$ in (1.1) yields (1.4). In summary:

The following expressions are all solutions to equation (1.1), conditional on the infinite sums be convergent:

$$A_t = A + \sum_{1}^{t} q_k$$

$$A_t = A + \sum_{-\infty}^{t} q_k \quad \text{“backward solution”}$$

$$A_t = A - \sum_{t+1}^{\infty} q_k \quad \text{“forward solution”}$$

2. First order difference equations

Consider the first order difference equation

$$x_t = ax_{t-1} + h_t \tag{2.1}$$

where x_t is an unknown sequence, $a \neq 0$ a constant and h_t a known sequence. Define a new sequence A_t by

$$x_t = a^t A_t$$

Inserting into (2.1) gives, after some simplification,

$$A_t - A_{t-1} = a^{-t} h_t \tag{2.2}$$

which is an equation of the type studied in the previous section.

In particular, assume that $h_t = h$ for all t, i.e., h_t is a constant independent of t. If $a \neq 1$, we can solve (2.2) by summing a geometric series:

$$A_t = A + \sum_{1}^{t} a^{-k} h = A + \frac{a^{-1} - a^{-t-1}}{1 - a^{-1}} h = A + \frac{1 - a^{-t}}{a - 1} h$$

i.e.,

$$x_t = A_t a^t = \left(A + \frac{h}{a-1} \right) a^t + \frac{h}{1-a} = Ba^t + \frac{h}{1-a}$$

where B is an arbitrary constant.

3. An example: Dynamic and intertemporal budget constraints

Let w_t be an individual's wage income in period t, $t = 0, 1, \ldots$, c_t her consumption and a_t her net assets (savings minus debt). Her *dynamic* budget constraint is then

$$a_t = a_{t-1} + ra_{t-1} + w_t - c_t$$

where r is the interest rate. We write this as

$$a_t = Ra_t + w_t - c_t$$

where $R \equiv 1 + r > 1$. The solution is $a_t = A_t R^t$, where

$$A_t - A_{t-1} = R^{-t}(w_t - c_t)$$

We solve this forward:

$$A_t = A - \sum_{k=t+1}^{\infty} R^{-k}(w_k - c_k)$$

which implies

$$a_t R^{-t} = A - \sum_{k=t+1}^{\infty} R^{-k}(w_k - c_k) \tag{3.1}$$

We assume that the individual is not permitted to let her debt grow indefinitely at a rate R or more: $\lim_{t\to\infty} a_t R^{-t} \geq 0$; this is called a *no Ponzi-game condition.* Nor, do we assume, does the individual want to for ever build up a positive wealth, at a rate R or more, which is never consumed: $\lim_{t\to\infty} a_t R^{-t} \leq 0$; hence $\lim_{t\to\infty} a_t R^{-t} = 0$. Letting $t \to \infty$ in (3.1) thus gives $A = 0$, i.e.,

$$a_t = - \sum_{k=t+1}^{\infty} R^{t-k}(w_k - c_k)$$

or

$$\sum_{k=t+1}^{\infty} R^{t-k} c_k = a_t + \sum_{k=t+1}^{\infty} R^{t-k} w_k \tag{3.2}$$

or in words: at any point in time, the present value of consumption equals the present value of future wage incomes plus the current net wealth. The constraint (3.2) is the individual's *intertemporal* budget constraint.

4. CAPM; another example

You will eventually read more about this in the macro course; Blanchard and Fischer's book chapter 6.

Let V_t be the value of a firm as of time t, and let π_t be its cash flow. $u(c_t)$ is the utility of consuming c_t for a representative consumer, and θ is the subjective discount rate, i.e., the total utility of as of now of consuming c_t this period and c_{t+1} next period is $u(c_t) + (1+\theta)^{-1}u(c_{t+1})$. The value of the firm then satisfies

$$(V_t + \pi_t)u'(c_t)(1+\theta)^{-1} = V_{t-1}u'(c_{t-1}) \tag{4.1}$$

To prove this, consider a consumer who has optimized her consumption path, and who considers buying a small portion δ of the firm this period $t-1$, refraining from consumption of equal value, and then selling it off in the next period t in order to increase her consumption then. The utility gain would be

$$-\delta V_{t-1}u'(c_{t-1}) + (1+\theta)^{-1}\delta(V_t + \pi_t)u'(c_t)$$

Obviously this must be ≤ 0; but this must be true also if δ is negative—she can sell of some today, etc.—, hence the expression above must be $= 0$, which proves (4.1).

Introduce the notation $z_t = u'(c_t)V_t$. We get

$$z_t = (1+\theta)z_{t-1} - \pi_t u'(c_t)$$

The general solution is $z_t = A_t(1+\theta)^t$ where A_t is a solution to

$$A_t - A_{t-1} = -(1+\theta)^{-t}\pi_t u'(c_t)$$

We solve this equation forward:

$$A_t = A + \sum_{k=t+1}^{\infty}(1+\theta)^{-k}\pi_k u'(c_k)$$

i.e.,

$$z_t = A_t(1+\theta)^t = A(1+\theta)^t + \sum_{k=t+1}^{\infty}(1+\theta)^{t-k}\pi_k u'(c_k)$$

and finally

$$V_t = \frac{z_t}{u'(c_t)} = \frac{A}{c'(c_t)}(1+\theta)^t + \sum_{k=t+1}^{\infty}(1+\theta)^{t-k}\frac{u'(c_k)}{c'(c_t)}\pi_k$$

The first term in the RHS is considered a *"bubble"* and is (rightly?) assumed zero (i.e., the constant $A = 0$,) furthermore the future is usually uncertain, so the final formula contains expectations:

$$V_t = \sum_{k=t+1}^{\infty}(1+\theta)^{t-k}E_t\left[\frac{u'(c_k)}{c'(c_t)}\pi_k\right]$$

5. Particular solution to second order difference equations

In Sydsæter and Hammond's book, it is shown how to find the general solution to the homogeneous difference equation

$$x_{t+2} + ax_{t+1} + bx_t = 0 \tag{5.1}$$

i.e., we can construct a solution x_t with arbitrary initial conditions $x_0 = d_0$, $x_1 = d_1$.

Let z_t be the solution to (5.1) which satisfies $z_0 = 0$, $z_1 = 1$. We can now construct a particular solution u_t^* to the *inhomogeneous* equation

$$x_{t+2} + ax_{t+1} + bx_t = c_t \tag{5.2}$$

A particular solution u_t^* to (5.2) is given by

$$u_t^* = \sum_{k=1}^{t} z_{t-k} c_{k-1}$$

This solution satisfies $u_0^* = u_1^* = 0$.

Proof:

$$\begin{aligned} u_{t+2}^* + au_{t+1}^* + bu_t^* &= \sum_{k=1}^{t+2} z_{t+2-k} c_{k-1} + a \sum_{k=1}^{t+1} z_{t+1-k} c_{k-1} + b \sum_{k=1}^{t} z_{t-k} c_{k-1} \\ &= \sum_{k=1}^{t} (z_{t+2-k} + az_{t+1-k} + bz_{t-k}) c_{k-1} + z_0 c_{t+1} + z_1 c_t + az_0 c_t \\ &= \dots \end{aligned}$$

here the first sum vanishes, since $z_{t+2-k} + az_{t+1-k} + bz_{t-k} = 0$, since z_t is a solution to the homogeneous equation (5.1). hence we can continue:

$$\begin{aligned} \dots &= 0c_{t+1} + 1c_t + a\,0\,c_t \\ &= c_t \end{aligned}$$

which proves that u_t^* is indeed a particular solution to (5.2). *Q.E.D.*

Also in this case we can use backward and forward solutions if appropriate for the application.

Example: solve the equation $x_{t+2} - 3.5x_{t+1} + 1.5x_t = c_t$, where c_t is some bounded sequence.

The roots of the characteristic equation is $r_1 = 3$ and $r_2 = 0.5$. The general solution to the corresponding homogeneous equation is thus $x_t^h = A\,3^t + B\,0.5^t$. We choose constants A and B so as to get a solution z_t satisfying $z_0 = 0$, $z_1 = 1$; we get

$z_t = 0.4\ 3^t - 0.4\ 0.5^t$, and hence the general solution to the non-homogeneous equation is

$$
\begin{aligned}
x^t &= 3^t A + 0.5^t B + \sum_{k=1}^{t} (0.4\ 3^{t-k} - 0.4\ 0.5^{t-k}) c_{k-1} \\
&= 3^t A + 0.5^t B + 0.4 \sum_{k=1}^{t} 3^{t-k} c_{k-1} - 0.4 \sum_{k=1}^{t} 0.5^{t-k} c_{k-1}
\end{aligned}
$$

If we want to solve the first sum forward and the second backwards, we replace A by $(A - 0.4 \sum_1^\infty 3^{-k} c_{k-1})$ and B by $(B - 0.4 \sum_{-\infty}^0 0.5^{-k} c_{k-1})$:

$$
\begin{aligned}
x^t &= 3^t (A - 0.4 \sum_{1}^{\infty} 3^{-k} c_{k-1}) + 0.5^t (B - 0.4 \sum_{-\infty}^{0} 0.5^{-k} c_{k-1}) \\
&\quad + 0.4 \sum_{k=1}^{t} 3^{t-k} c_{k-1} - 0.4 \sum_{k=1}^{t} 0.5^{t-k} c_{k-1} \\
&= 3^t A + 0.5^t B - 0.4 \sum_{k=t+1}^{\infty} 3^{t-k} c_{k-1} - 0.4 \sum_{k=-\infty}^{t} 0.5^{t-k} c_{k-1}
\end{aligned}
$$