Exam MATFUII 29/3 1995

Clearly state all steps towards the answer. Using a correct method is more important than getting all the algebra exactly correct. Calculators not capable of deriving analytical solutions to differential and/or difference equations are allowed but hardly useful. No other aid is allowed. If you make any assumptions, state them clearly.

Each of the four question gives 25 points. To pass you need at least 6 points on each and a minimum sum of 50 points.

Good luck!

1.

Solve the following differential equations

a) $\dot{y}(t) - 2ty(t) = 1$

b) $\dfrac{\dot{y}(t) - y(t)}{t^2} = 1$

c) $\ddot{y}(t) + 2\dot{y}(t) - 3y(t) = 6t^2 - t - 1,$

d) $\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -x_1(t) - x_2(t) \\ -x_1(t) + x_2(t) \end{bmatrix}$

2.

Solve the following difference equations, make use of initial conditions when provided.

$a)$ $y_{t+2} + 3y_{t+1} + 2y_t = 3$

$b)$ $y_{t+3} - 9y_{t+2} + 27y_{t+1} - 27y_t = -2$

$c)$ $y_{t+2} + 2y_{t+1} - 3y_t = 12,$
$y_1 = 1, y_2 = 2$

$d)$ $\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix},$
$x_0 = 0, x_1 = 1$

Hint for b); expand $(r-3)^3$.
Hint for *d*); the eigenvectors are identical, transform the system into one difference equation.

3.

Consider the following consumption problem. The consumer gets utility from consuming c_t which costs p_t per unit. She gets an income flow of $f(k)$ and the capital stock depreciates at rate δ. f and U have the usual concavity properties. The consumer thus solves

$$\max_{\{c_t\}} \int_0^{\infty} e^{-rt} U(c_t)\,dt$$

$$\begin{aligned} s.t. \quad & \dot{k}_t = f(k) - p_t c_t - \delta k_t, \\ & k_0 = 0, \\ & k_t \geq 0. \end{aligned}$$

a) Write the current value Hamiltonian and the necessary conditions for an optimum.

b) Take time derivatives of the condition involving the derivative of the Hamiltonian with respect to the control variable. Note that also the price is changing over time. Use this to get rid of the shadow value in the condition involving the derivative of the Hamiltonian with respect to the state variable. State your result in an equation for $\dot{c}$. Interpret the equation, what happens if inflation is high vs. low?

c) Now assume $\dot{p}_t = 0, \forall t$ and that the U is of CARA-type. Draw a phase diagram of the system with k on the x-axis and c on the y-axis. Be careful when you draw the arrows. If you make further assumptions, state them clearly.

d) Mark the solution to the problem.

e) Let the system be in a steady state. Assume that there is an unexpected jump upwards in the price level. After that $\dot{p}_t = 0, \forall t$. Show how k and c responds to the shock. Indicate potential jumps as well as gradual adjustments.

4.

Define the value function

$$
\begin{aligned}
V(k) &= \max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t U(c_t) \\
s.t. \quad & k_{t+1} = f(k_t) - c_t, \\
& k_0 = k, \\
& k_t \geq 0 \quad \forall t.
\end{aligned} \tag{4.1}
$$

a) Substitute from the dynamic budget constraint to get rid of c_t. Write the Bellman equation as a maximum over k_{t+1} with the first order condition for a maximum. Make sure all time subscripts are correct.

$$
\begin{aligned}
&V(k_t) = \max_{k_{t+1}} \left[U\left(f(k_t) - k_{t+1} \right) + \beta V\left(k_{t+1}\right) \right] \\
&FOC; \quad -U'\left(f(k_t) - k_{t+1} \right) + \beta V'\left(k_{t+1}\right) = 0.
\end{aligned}
$$

b) Use the envelope theorem to calculate $V'(k_t)$ and use the result to get rid of $V'(\cdot)$ in your answer to a).

Now let U and f be given by

$$
\begin{aligned}
&U(c_t) = \ln c_t \\
&f(k_t) = k_t^{\alpha} \\
&0 < \alpha < 1.
\end{aligned}
$$

d) Show that $V(k_t) = A \ln k_t + B$ solves the Bellman equation.

e) Solve for A and the optimal consumption rule in terms of parameters and k_t.

f) Now assume there is habit persistence so the per period utility function in (4.1) is

$$U = U(c_t, c_{t-1})$$

Write the Bellman equation with its associated first order condition.