Exam MATFUII 8/1 1996

The exam consists of two parts. The first is written by John Hassler and contains 3 main questions. The second is written by Tomas Björk and contains 2 main questions. Each of the 5 main questions have a maximum of 20 points. To pass you must

- get a total score of at least 50 points, and
- get at least 6 points on *each of the five questions.*

Please note that you must satisfy both these requirements to pass. So make sure that you answer all five questions at least partially.

All calculators not capable of deriving analytical solutions to differential and/or difference equations are allowed but hardly useful. No other aid is allowed.

Good luck!

Part 1 by John Hassler

Clearly state all steps towards the answer. Using a correct method is more important than getting all the algebra exactly correct. If you make any assumptions, state them clearly.

1.

Solve the following differential and difference equations. Use initial conditions when given..

a) $$\frac{\dot{y}(t)t}{3y(t)}=1.$$

b) $$\ddot{y}(t)+2\dot{y}(t)+y(t)=e^t.$$

c) $$y_{t+2}+2y_{t+1}-3y_t=12,$$
$$y_1=1, y_2=2.$$

d) $$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\begin{bmatrix}3x_1(t)-2x_2(t)\\ 2x_1(t)-2x_2(t)\end{bmatrix},$$
$$\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix}.$$

e) $$\begin{bmatrix}x_{t+1}\\ y_{t+1}\end{bmatrix}=\begin{bmatrix}3x_t-2y_t\\ 2x_t-2y_t\end{bmatrix}+\begin{bmatrix}1\\ 1\end{bmatrix}.$$

Preliminary answer

a)
$$g(t)^{-1}=y^{-1}$$
$$G(t)=\ln y=3\int t^{-1}dt+c=3\ln t+c$$
$$y=e^c t^3.$$

b)
$$r_{1,2}=-1.$$
$$y_h=c_1e^{-t}+c_2te^{-t}.$$
$$\text{guess } y_p=ke^t$$
$$\Rightarrow ke^t+2ke^t+ke^t=e^t$$
$$k=1/4.$$
$$y=c_1e^{-t}+c_2te^{-t}+e^t/4$$

c)

$$\begin{aligned}
&r^2+2r-3=0 \Rightarrow r=1,-3\\
&y_t^h=c_1+c_2(-3)^t\\
&\text{guess } y_t^p=kt\\
&\Rightarrow k(t+2)+2k(t+1)-3kt=4k=12\\
&\Rightarrow k=3\\
&y_1=c_1+c_2(-3)^1+3=1 \Rightarrow c_1=3c_2-2.\\
&y_2=3c_2-2+c_2(-3)^2+6=2\\
&\Rightarrow c_2=-1/6, c_1=-5/2.
\end{aligned}$$

d

2.

Consider the following simple variant of a dynamic IS-LM model with fixed prices. In this economy there is a short (over night) interest rate denoted *i*. There also exist (infinitely) long bonds that pay \$1 per unit of time. The direct return *R* from these bonds are 1 over their price. Total return is *R* plus the rate of change in bond prices. This equals $R-\dot{R}/R$. Aggregate demand A is a function of long bond rates *R* and output *Y*. Output adjusts slowly to the equilibrium where demand equals output. This produces a differential equation for output

$$\begin{aligned}
\dot{Y}&=\phi[A(R,Y)-Y]\\
&\phi>0, A_R<0,\\
&1>A_Y>0.
\end{aligned} \tag{1}$$

The LM curve gives us the short interest rate *i* as a function of the fixed money stock *M* and output

$$\begin{aligned}
&i=L(M,Y)\\
&L_M<0, L_Y>0.
\end{aligned} \tag{2}$$

Now assume that arbitrage in the financial markets assure that the total return on bonds equals short rates.

$$i=R-\dot{R}/R \tag{3}$$

a) Use the equations (1)-(3) to derive a system of differential equations for $\dot{R}$ and $\dot{Y}$.
b) Calculate the signs of the slopes dR/dY along the curves given by $\dot{R}$=0 and $\dot{Y}$=0.
c) Draw the phase diagram for the system. Is the system stable, unstable or saddle path stable?
d) Assume the system is in a steady state. What happens to *R*, *Y* and *i* over time if *M* unexpectedly increases?

Preliminary answers

a)

$$\begin{aligned}
\dot{Y}&=\phi[A(R,Y)-Y]\\
\dot{R}&=R[R-L(M,Y)]
\end{aligned}$$

b)

$$\dot{Y} = 0 \Rightarrow Y = A(R,Y)$$
$$\Rightarrow \frac{dR}{dY} = \frac{1-A_Y}{A_R} < 0.$$
$$\dot{R} = 0 \Rightarrow R = L(M,Y)$$
$$\frac{dR}{dY} = L_Y > 0.$$

3.

Assume a consumer solves the following problem

$$\begin{aligned} &\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t U(c_t) \\ &s.t. \qquad k_{t+1} = f(k_t) - c_t, \\ &\qquad\qquad k_0 = k, \\ &\qquad\qquad k_t \geq 0 \quad \forall t. \end{aligned} \tag{1}$$

a) Substitute from the dynamic budget constraint to get rid of c_t. Write the Bellman equation as a maximum over k_{t+1} with the first order condition for a maximum. Make sure all time subscripts are correct.

$$V(k_t) = \max_{k_{t+1}} \left[U\left(f(k_t) - k_{t+1} \right) + \beta V\left(k_{t+1}\right) \right]$$
$$FOC; \quad -U'\left(f(k_t) - k_{t+1} \right) + \beta V'\left(k_{t+1}\right) = 0.$$

Now let U and f be given by

$$\begin{aligned} &U(c_t) = \ln c_t \\ &f(k_t) = k_t^{\alpha} \\ &0 < \alpha < 1. \end{aligned} \tag{2}$$

b) Show that the value function $V(k_t) = A \ln k_t + B$ solves the Bellman equation.

c) Solve for completely for the path of k_t.(Hint; solve in terms of $\log k_t$). Then solve for the optimal consumption path.

d) Now assume there is habit persistence so the per period utility function in (1) is

$$U = U(c_t, c_{t-1}) \tag{3}$$

Write the Bellman equation with its associated first order condition. (Hint; the value function at t must have arguments that fully determine how much utility can be reached from t and onwards.)

Answer

The possibly most natural formulation is

$$V(k_t, c_{t-1}) = \max_{k_{t+1}} U(c_t, c_{t-1}) + \beta V(k_t, c_{t-1})$$

First order conditions are

$$-U_1^t + \beta\left(V_1^{t+1} - V_2^{t+1}\right) = 0$$

Using the envelope theorem this can be written

$$U_1^t + \beta U_2^{t+1} = \beta f_1^{t+1}\left(U_1^{t+1} + U_2^{t+2}\right)$$