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EXAM
MatFuII
5 points

Exam MATFUII 3/4 1996

The exam consists of two parts. The first is written by John Hassler and contains 3 main questions. The second is written by Tomas Björk and contains 2 main questions. Each of the 5 main questions have a maximum of 20 points. To pass you must

- get a total score of at least 50 points, and
- get at least 6 points on each of the three questions in John Hassler's part and at least a total of 12 points on Tomas Björk's part.

Please note that you must satisfy both these requirements to pass. So make sure that you answer all five questions at least partially.

All calculators not capable of deriving analytical solutions to differential and/or difference equations are allowed but hardly useful. No other aid is allowed.

Good luck!

1. Difference and Differential Equations

Solve the following differential and difference equations. Use initial conditions when given.

a) $$\frac{\dot{y}(t) - y(t)}{t} = 1.$$

b) $$\dot{y}(t) - \frac{y(t)}{t} = 1.$$

c) $$\ddot{y}(t) + 2\dot{y}(t) - 3y(t) = 6t^2 - t - 1.$$

d) $$y_{t+2} + 2y_{t+1} - 3y_t = 12,$$ $$y_1 = 1, y_2 = 2.$$

e) $$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix},$$ $$x_0 = 0, x_1 = 1.$$

2. Dynamic Optimization in Continuous Time

Consider the following consumption problem. The consumer gets utility from consuming c_t which costs p_t per unit. Prices are not necessarily constant so goods inflation may be non-zero but finite. She gets an income flow of $f(k_t)$ and the capital stock depreciates at rate δ. f and U have the usual concavity properties. The consumer thus solves

$$\max_{\{c_t\}} \int_0^\infty e^{-rt} U(c_t) dt$$

$$\begin{aligned} s.t. \quad & \dot{k}_t = f(k_t) - p_t c_t - \delta k_t, \\ & k_0 = 0, \\ & k_t \geq 0. \end{aligned}$$

a) Write the current value Hamiltonian and the necessary conditions for an optimum.

b) Take time derivatives of the condition involving the derivative of the Hamiltonian with respect to the control variable. Note that also the price is changing over time. Use this to get rid of the shadow value in the condition involving the derivative of the Hamiltonian with respect to the state variable. State your result in an equation for $\dot{c}$. Interpret the equation, what happens if inflation is high vs. low?

c) Now assume $\dot{p}_t = 0, \forall t$ and that the U is of CARA-type. Draw a phase diagram of the system with k on the x-axis and c on the y-axis. Be careful when you draw the arrows. If you make further assumptions, state them clearly. Mark the full solution to the problem in your phase diagram.

d) Let the system be in a steady state. Assume that there is an unexpected jump upwards in the price level. After that $\dot{p}_t = 0, \forall t$. Show how k and c responds to the shock. Indicate potential jumps as well as gradual adjustments.

3. Dynamic Optimization in Discrete Time

Define the value function

$$
\begin{aligned}
&V(A) = \max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t U(c_t) \\
&s.t. \qquad A_{t+1} = RA_t - c_t, \\
&\qquad\qquad A_0 = A, \\
&\qquad\qquad A_t \geq 0 \quad \forall t.
\end{aligned}
$$

with parameters satisfying

$$
\begin{aligned}
&0 < R\beta^2 < 1, \\
&R > 1, \\
&0 < \beta < 1.
\end{aligned}
$$

a) Substitute from the dynamic budget constraint to get rid of c_t. Write the Bellman equation as a maximum over A_{t+1} with the first order condition for a maximum.

Preliminary answer

$$FOC; \quad -U'\left(RA_t - A_{t+1}\right) + \beta V'\left(A_{t+1}\right) = 0.$$

b) Let U be the CRRA function

$$U(c_t) = \frac{c_t^{1-1/2}}{1-1/2} = 2\sqrt{c_t}$$

and guess that $V(A) = 2\gamma\sqrt{A}$ for some constant γ. Use this to show that along the optimal path

$$A_{t+1} = \frac{(\beta\gamma)^2}{1+(\beta\gamma)^2} RA_t$$

Preliminary answer

Substitute into the FOC for the Bellman equation

$$
\begin{aligned}
&\left(RA_t - A_{t+1}\right)^{-1/2} = \beta\gamma\left(A_{t+1}\right)^{-1/2} \\
&\Rightarrow \left(\beta\gamma\right)^2\left(RA_t - A_{t+1}\right) = A_{t+1} \\
&\Rightarrow \frac{\left(\beta\gamma\right)^2}{1+\left(\beta\gamma\right)^2} RA_t = A_{t+1.}
\end{aligned}
$$

c) Verify that the guess was correct and show that

$$\gamma = \sqrt{\frac{R}{1-R\beta^2}}$$

Preliminary answer

Substitute the solution to the first order condition into the Bellman Equation

$$
\begin{aligned}
2\gamma A_t^{1/2} &= 2\left(RA_t - A_{t+1}\right)^{1/2} + 2\beta\gamma A_{t+1}^{1/2} \\
&= 2\left(RA_t - \frac{(\beta\gamma)^2}{1+(\beta\gamma)^2}RA_t\right)^{1/2} + 2\beta\gamma\left(\frac{(\beta\gamma)^2}{1+(\beta\gamma)^2}RA_t\right)^{1/2} \\
\gamma\sqrt{A_t} &= \sqrt{RA_t}\,\frac{1}{\left(1+(\beta\gamma)^2\right)^{1/2}} + \sqrt{RA_t}\,\frac{(\beta\gamma)^2}{\left(1+(\beta\gamma)^2\right)^{1/2}} \\
&= \sqrt{A_t}\sqrt{R\left(1+(\beta\gamma)^2\right)} \\
\Rightarrow \gamma^2 &= R\left(1+(\beta\gamma)^2\right) \\
\gamma^2 &= \frac{R}{\left(1-R\beta^2\right)} \\
\gamma &= \sqrt{\frac{R}{\left(1-R\beta^2\right)}}
\end{aligned}
$$

d) Show that the last constraint $A_t \geq 0 \quad \forall t$ is satisfied and solve for the path of c_t.

Preliminary answer

Using the expression for γ€in the transition equation for A_t we get

$$
\begin{aligned}
A_{t+1} &= \frac{(\beta\gamma)^2}{1+(\beta\gamma)^2}RA_t \\
&= \frac{\dfrac{\beta^2 R}{\left(1-R\beta^2\right)}}{1+\dfrac{\beta^2 R}{\left(1-R\beta^2\right)}}RA_t = \frac{\dfrac{\beta^2 R}{\left(1-R\beta^2\right)}}{\dfrac{1}{\left(1-R\beta^2\right)}} = \beta^2 R^2 A_t
\end{aligned}
$$

Solving this difference equation we have

$$A_t = (R\beta)^{2t} A$$

which is positive for all t.

Using the budget constraint we have that

$$
\begin{aligned}
c_t &= RA_t - A_{t+1} \\
&= RA_t - \beta^2 R^2 A_t \\
&= R\left(1-R\beta^2\right)A_t \\
&= R\left(1-R\beta^2\right)(R\beta)^{2t} A
\end{aligned}
$$