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August 26, 1996

EXAM
MatFuII
5 points

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The exam consists of two parts. The first is written by John Hassler and contains 3 main questions. The second is written by Tomas Björk and contains 2 main questions. Each of the 5 main questions have a maximum of 20 points. To pass you must

- get a total score of at least 50 points, and
- get at least 6 points on each of the three questions in John Hassler's part and at least a total of 12 points on Tomas Björk's part.

Please note that you must satisfy both these requirements to pass. So make sure that you answer all five questions at least partially.

All calculators not capable of deriving analytical solutions to differential and/or difference equations are allowed but hardly useful. No other aid is allowed.

Good luck!

1. Difference and Differential Equations

Solve

a) $$\dot{x}(t)x(t)^{-2} = 1/t.$$

b) $$\frac{\dot{y}(t) - y(t)}{t} = 1$$

c) $$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

d) $$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

e) $$y_{t+2} + y_{t+1} - 2y_t = 12$$

f) Find a forward or backward solution to

$$\begin{aligned} &A_t - A_{t-1} = q_t, \\ &q_t = 1.1 q_{t-1}, \qquad . \\ &q_0 = 1. \end{aligned}$$

2. Dynamic Optimization in Continuous Time

Consider the following investment problem.

$$\begin{aligned} &\max_i \int_0^\infty e^{-rt}\left(f(k) - i(p + c(i))\right)dt \\ &s.t. \qquad \dot{k} = i - \delta k \\ &\qquad\qquad k_0 = 0 \end{aligned}$$

where $f(k)$ is a production function, p is a (constant) price of investment goods, i is investments and $c(i)$ is an installation cost. Now assume

$$\begin{aligned} &f(k) = \alpha k - \frac{\beta k^2}{2}, \\ &c(i) = \frac{\gamma i}{2} \end{aligned}$$

and that all parameters are strictly positive.

a) Write the current value Hamiltonian and the necessary conditions for an optimum.

b) Use the condition involving the derivative of the Hamiltonian with respect to the control variable to substitute for *i*. State your result in a system of linear differential equations for the current shadow value q_t and k_t.

c) Draw a phase diagram of the system with *k* on the *x*-axis and *q* on the *y*-axis. Be careful when you draw the arrows. Indicate the optimal paths for k_t and q_t when the horizon is finite and equal to *T* if the terminal condition is that k_T is zero.

d) Assume *T* is infinite and the solution settles down to a steady state. Mark the optimal path k_t and q_t solution to the problem.

e) Let the system be in a steady state. Assume that there is an unexpected permanent increase in marginal productivity by an upwards shift in α. Show what happens dynamically.

f) Which parameters determine the speed of convergence to the steady state.

3. Dynamic Optimization in Discrete Time

Consider an individual who derives utility from a stock of durables that depreciates at the constant and positive rate δ. The individuals has access to a perfect capital market at which he can invest or borrow at a positive interest rate *r*. His purpose in life is to maximize his discounted sum of present and future utility. Let A_t be his financial assets and W_t be the total assets before making any purchases so that $W_t = A_t + (1-\delta)k_{t-1}$. Let the value function be given by

$$\begin{aligned} V(W_t) &= \max_{\{k_s\}_t^\infty} \sum_{s=0}^{\infty} \beta^s U(k_{t+s}) \\ s.t. \quad & W_{t+1} = W_t(1+r) - k_t(r+\delta), \\ & W_t \geq 0 \quad \forall t. \end{aligned}$$

a) Write the Bellman equation for the problem and its associated first order condition for k_t.

b) Use the envelope theorem to find a relation between marginal utilities in subsequent periods.

c) Assume that $U(k_t) = ln(k_t)$. Use your answer to *b*) to derive a difference equation for k_t and find its set of solutions.

d) Guess a form of *V* and verify that your guess is correct by showing that it satisfies the Bellman equation.

e) Find the consumption function $c(W_t)$ which gives c_t as a function of W_t and the problem parameters.