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EXAM

MatFuII

5 points

The exam consists of 6 questions which in total give a maximum of 100 points. The maximum score on each question is provided together with the question. Use this information to allocate your time wisely. To pass you must get a total score of at least 60 points.

All calculators not capable of deriving analytical solutions to differential and/or difference equations are allowed but hardly useful. No other aid is allowed.

Good luck and Merry Christmas!

John Hassler

1. (10 points)

Solve completely the differential equation

$$y'(t) + \frac{1}{t} y(t) = 3t \, .$$

2. (12 points)

Solve the following system of difference equations

$$\begin{aligned} x_{t+1} &= 1.1x_t + 0.9y_t - 1 \\ y_{t+1} &= 0.9x_t + 1.1y_t - 1 \end{aligned}$$

3. (18 points)

a) Solve the following system of differential equations

$$\begin{aligned} x'(t) &= 0.4x(t) - 0.2y(t) - 0.2 \\ y'(t) &= 0.7x(t) - 0.5y(t) - 0.2 \end{aligned}$$

b) Draw the phase diagram, where the saddle path is clearly marked, and give the slope of the saddle path.

4. Optimal Control (25 points)

Consider the following consumption problem

$$\begin{aligned} &\max_{\{c_t\}_0^\infty} \left(\int_0^\infty \left(\frac{c_t^{\,1-\sigma}}{1-\sigma} \right) e^{-\rho t} dt \right) \quad \sigma > 0 \\ &s.t \qquad \dot{k}_t = (1-\tau) k_t^{\,1-\alpha} g_t^{\,\alpha} - c_t \\ &\qquad\qquad k_0 = \bar{k} > 0 \\ &\qquad\qquad k_t > 0 \forall t. \end{aligned}$$

where c is consumption, k is a capital stock g is government spending and σ, ρ, r and α are parameters of the problem. τ is a fixed tax rate.

a) Set up the current value Hamiltonian, denoted $\mathcal{H}(\ )$ with necessary conditions for optimum. Define the multiplier (shadow value) as μ.

b) Recall that $d \ln c / dt = \dot{c}/c \equiv \hat{c}$. Now use the FOC for a maximum of the Hamiltonian to get an expression for $\hat{c}$ in terms of $\hat{\mu}$. (Take logs and differentiate with respect to time).

c) Use your answer to *b* to get an expression for $\hat{c}$ in terms of *g, k,* τ and parameters from the necessary condition for $\mathcal{H}_k$. (Hint; first divide the condition for $\mathcal{H}_k$. by μ).

d) Now assume that the government spending on g is financed via a flat and constant income tax so $g = \tau k^{1-a} g^{\alpha}$. Use this to eliminate g and k in the expression for $\hat{c}$. You can then write $\hat{c}$ as a function of σ, τ, α and ρ.

e) Now find the central planner problem. Substitute from the government budget constraint into the transition equation for k and set up the Hamiltonian, letting the government choose both the consumption path and the optimal tax rate. What is the optimal tax rate τ and the optimal growth rate of consumption?

5. Dynamic Programming (20 points)

Consider a consumer who maximizes a CARA consumption function and has access to a perfect capital market with an interest rate that equals his subjective discount rate. He solves

$$\begin{aligned}
&\max_{\{c_t\}} \sum_{t=0}^{\infty} -(1+r)^{-t} e^{-\gamma c_t} \\
&s.t. \qquad A_{t+1} = (1+r)(A_t - c_t), \\
&\qquad\qquad A_0 = A, \\
&\qquad\qquad \lim_{T\to\infty} A^T (1+r)^{-T} \geq 0.
\end{aligned} \tag{1}$$

where c is consumption, A are financial assets and r and γ are parameters of the problem.

a) Write down the Bellman equation. Make sure time subscripts and what is maximized over is clearly stated.

b) Write down the FOC condition for an interior maximum of the Bellman equation.

c) Guess that the value function is given by

$$V(k) \equiv -\frac{1+r}{r} e^{-\gamma \frac{r}{1+r} k}$$

Is this the correct value function? Prove your statement.

6. Numerical Solution to Differential Equations (15 points)

Let **x** be a 2x1 vector with the elements x_1 and x_2. Assume we want to solve the following set of equations with the Newton-Raphson method.

$$
\begin{aligned}
f_1(x_1, x_2) &= 1, \\
f_2(x_1, x_2) &= 1.
\end{aligned}
$$

a) Write down the equation that defines the Newton-Raphson algorithm for this problem. (Hint; Use a first order Taylor approximation and note that the RHS of the equation is **1** not **0**.)

b) Calculate the updated value $\mathbf{x}^1$ if we start at $\mathbf{x}^0 = \{0,0\}$ when we set

$$
\begin{aligned}
f_1(x_1, x_2) &= x_1 e^{x_2}, \\
f_2(x_1, x_2) &= x_2 e^{2x_1}.
\end{aligned}
$$

Answer

The current value Hamiltonian and its optimality conditions are:

$$
\begin{aligned}
&H = \frac{c^{1-\sigma}}{1-\sigma} + \mu\left((1-\tau)k^{1-\alpha}g^{\alpha} - c\right) \\
&H_c = c^{-\sigma} - \mu = 0 \Rightarrow \hat{c} = -\frac{\hat{\mu}}{\sigma} \\
&H_k = \mu(1-\alpha)(1-\tau)\left(\frac{g}{k}\right)^{\alpha} = -\dot{\mu} + \rho\mu \Rightarrow -\hat{\mu} = (1-\alpha)(1-\tau)\left(\frac{g}{k}\right)^{\alpha} - \rho \\
&\Rightarrow \hat{c} = \sigma^{-1}\left((1-\alpha)(1-\tau)\left(\frac{g}{k}\right)^{\alpha} - \rho\right) = \sigma^{-1}\left(\mathrm{MPK}_{\text{private}} - \rho\right)
\end{aligned}
$$

Using the production function and the governments budget constraint we get:

$$
\begin{aligned}
&\frac{k^{1-\alpha}g^{\alpha}}{g} = \left(\frac{k}{g}\right)^{1-\alpha} \Rightarrow \tau = \frac{g}{y} = \left(\frac{g}{k}\right)^{1-\alpha} \\
&\Rightarrow \gamma = \sigma^{-1}\left((1-\alpha)(1-\tau)\tau^{\frac{\alpha}{1-\alpha}} - \rho\right)
\end{aligned}
$$

By doing the same substitution as in we can write the problem of the government as.

$$\max_{c,\tau}\left(\int_0^\infty \left(\frac{c_t^{\,1-\sigma}}{1-\sigma}\right)e^{-\rho t}\,dt\right) \quad \sigma>0$$

$$\begin{aligned} s.t \qquad & \dot{k} = (1-\tau)\tau^{\alpha/(1-\alpha)}k - c \\ & \lim_{T\to\infty} k_T e^{-rT} = 0 \end{aligned}$$

The Hamiltonian and its optimality conditions are:

$$H = \frac{c^{1-\sigma}}{1-\sigma} + \mu\left((1-\tau)\tau^{\frac{\alpha}{1-\alpha}}k - c\right)$$

$$H_c = c^{-\sigma} - \mu = 0 \Rightarrow \hat{c} = -\frac{\hat{\mu}}{\sigma}$$

$$H_\tau = \mu k\left(\frac{\alpha}{1-\alpha}(1-\tau)\tau^{\frac{\alpha}{1-\alpha}}\tau^{-1} - \tau^{\frac{\alpha}{1-\alpha}}\right) = \mu k\tau^{\frac{\alpha}{1-\alpha}}\left(\frac{\alpha}{1-\alpha}\frac{1-\tau}{\tau} - 1\right) = 0 \Rightarrow \tau = \alpha$$

$$H_k = \mu(1-\tau)\tau^{\frac{\alpha}{1-\alpha}} = -\dot{\mu} + \rho\mu \Rightarrow -\hat{\mu} = (1-\tau)\tau^{\frac{\alpha}{1-\alpha}} - \rho$$

$$\Rightarrow \hat{c} = \sigma^{-1}\left((1-\tau)\tau^{\frac{\alpha}{1-\alpha}} - \rho\right) = \sigma^{-1}\left(\mathrm{MPK}_{\mathrm{social}} - \rho\right) = \sigma^{-1}\left((1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} - \rho\right)$$