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# EXAM

# MatFuII

# 5 points

The exam consists of 5 questions, which in total give a maximum of 100 points. The maximum score on each question is provided together with the question. Use this information to allocate your time wisely. To pass you must get a total score of at least 60 points.

All calculators not capable of deriving analytical solutions to differential and/or difference equations are allowed but hardly useful. No other aid is allowed.

## 1. Differential and difference equations (25 points)

Solve the following differential and difference equations. Use initial conditions when given.

a) $\ddot{y}(t)+2\dot{y}(t)+y(t)=e^{t}$ .

b) $\dot{x}=x^{2}t^{2}$

c) $y_{t+2}+2y_{t+1}-3y_{t}=12,$
$y_1=1, y_2=2.$

d) $\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\begin{bmatrix}3x_1(t)-2x_2(t)\\ 2x_1(t)-2x_2(t)\end{bmatrix},$
$\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix}.$

e) $\begin{bmatrix}x_{t+1}\\ y_{t+1}\end{bmatrix}=\begin{bmatrix}-1 & 1\\ -4 & 3\end{bmatrix}\begin{bmatrix}x_t\\ y_t\end{bmatrix}$
$\begin{bmatrix}x_0\\ x_1\end{bmatrix}=\begin{bmatrix}0\\ 1\end{bmatrix}.$

Hints:

a) Guess on a particular solution of similar form as the RHS.

e) Use the first equation to eliminate *y* in the second equation, yielding a second order difference equation in *x*.

## 2. Phase diagrams (15 points)

Consider the solution to the Ramsey consumption problem (you can forget that this is an economical example, however)

$$\dot{c}(t)=\sigma\left(f'(k)-\theta\right)$$
$$\dot{k}(t)=f(k)+T-c$$

where *c* is consumption, $\sigma$ is a positive constant, *f* is a standard production function, (i.e., *f*'>0 and *f*''<0), *k* is capital and *T* is a transfer.

a) Draw the phase diagram, make sure you indicate the laws of motion in all quadrants and at the lines where time derivatives in one dimension are zero and draw the paths to steady state(s).

b) Consider a sudden positive and permanent shift in $T$ that occurs when the system is in steady state. Describe the path to the new steady state in the phase diagram.

## 3. Optimal Control (25 points)

Consider the problem of investing in a stock of capital over a fixed time horizon. The flow of profits is given by $P(x)$ where $P$ is a concave function which we will specify below and $x$ is the current capital stock. Investment costs $C(u)$ are increasing and convex in the in the investment rate, denoted $u$. The capital stock depreciates at rate $\delta$.The problem is to maximize the present value of the profit stream from 0 to $T$, applying a discount rate of $r$.

$$
\begin{aligned}
&\max_{u(t)} \left( \int_0^T \left( P(x) - C(u) \right) e^{-rt} dt \right) \\
&s.t \qquad \dot{x}(t) = u(t) - \delta x(t), \\
&\qquad\quad\;\; x(0) = x_0 > 0, \\
&\qquad\quad\;\; u(t) \geq 0.
\end{aligned} \tag{1}
$$

We assume, for simplicity that the non-negativity constraint is satisfied automatically so we can disregard it.

*a)* Set up the Hamiltonian, denoted $H(\ )$ with necessary conditions for optimum. Define the multiplier (shadow value) as $\lambda(t)$.

*b)* What is the terminal condition for $\lambda(T)$?

*c)* Show that investments at $t$ should be done so that current marginal cost of investments, $C'(u(t))$ equals the marginal productivity, $P'(x(t)$, integrated over the remaining horizon and discounted at rate $-(r+\delta)$. Interpret!. (Hint: Start with the condition involving $H_x$, move terms containing the shadow value to the LHS, multiply by the integrating factor, integrate over the remaining horizon and use the terminal condition for $\lambda(T)$?.

*d)* Now specify that $P(x)=x$ and $C(u)=u^2/2$. Find the optimal investment plan.

## 4. Dynamic Programming (25 points)

Consider a consumer who derives utility by holding a stock of durables according to a CARA utility function. The price per unit of the durable is unity and it depreciates at rate $\delta$. He has access to a perfect capital market with an interest rate that equals his subjective discount rate. He solves

$$
\begin{aligned}
&\max_{\{k_t\}} \sum_{t=0}^{\infty} -(1+r)^{-t} e^{-\gamma k_t} \\
&s.t. \qquad W_{t+1} = (1+r)(W_t - k_t) + (1-\delta)k_t, \\
&\qquad\qquad W_0 = \overline{W}, \\
&\qquad\qquad \lim_{T\to\infty} W^T (1+r)^{-T} \geq 0.
\end{aligned} \tag{2}
$$

where $k$ is the stock of the durable, $W$ denotes total assets of the individual, $r$ is the interest rate a $\gamma$ is the risk aversion parameter.

*a)* Write down the Bellman equation. Make sure time subscripts, arguments and what is maximized over is clearly and correctly stated.

*b)* Since the interest rate and the subjective discount rate coincide, we can guess that the stock of durables and wealth holdings are constant over time. Use this guess and the law of motion for $W$ to find $k$ as a function of $W$.

*c)* Substitute your answer from *b*) into the Bellman equation and find the value function.

*d)* Verify that your guess was correct.

## 5. Numerical Solution to Differential Equations (10 points)

Consider the differential equation

$$
\begin{aligned}
&y'(x) + ay(x) = x^2, \\
&y(1) = 0.
\end{aligned} \tag{3}
$$

Provide a numerical solution to (3) at $x$=1,2,3 and 4. Take the following steps.

a) Use forward differences to approximate the derivatives, i.e., replace $y'(x)$ by $y(x+1)-y(x)$.

b) Set up the initial condition and the approximations to (3) evaluated at the four values of $x$ as a system of linear equations for $y$ and $x$.

c) Express the system on matrix form. Find the solution for $a$=1.