# Exam MATFUII 11/1 1999

John Hassler

Clearly state all steps towards the answer. Using the right method is more important than getting all the algebra exactly correct. Calculators not capable of deriving analytical solutions to differential and/or difference equations are allowed but hardly useful. No other aid is allowed. If you need to make any assumptions, state them clearly.

To pass you need a minimum of 50 points. Note that the final questions on section 3 and 4 are starred. This implies that you are asked to slightly extend the methods we discussed in class. I suggest you do these questions last if you have time.

I will be available for questions approximately one hour after the start of the exam.

Good luck!

## 1. Differential Equations (15 points)

Solve completely the following differential equations, make use initial conditions when provided.

a) $\ddot{y}(t)+2\dot{y}(t)+y(t)=t^2, y(0)=0, y(1)=3$.

b) $\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\begin{bmatrix}x_1(t)+4x_2(t)\\ x_1(t)+x_2(t)\end{bmatrix}$.

Hint for b); A set of eigenvectors for the coefficient matrix is $\begin{bmatrix}1 & -1\\ 0.5 & 0.5\end{bmatrix}$ with an inverse $\begin{bmatrix}0.5 & 1\\ -0.5 & 1\end{bmatrix}$.

## 2. Difference Equations (15 points)

Solve the following difference equations, make use of initial conditions.

a) $y_{t+3}-6y_{t+2}+12y_{t+1}-8y_t=1, y_0=0, y_1=-1, y_2=-9$.

b) $\begin{bmatrix}x_{t+1}\\ y_{t+1}\end{bmatrix}=\begin{bmatrix}-1 & 1\\ -4 & 3\end{bmatrix}\begin{bmatrix}x_t\\ y_t\end{bmatrix}$,

$x_0=0, x_1=1.$

Hint for a); expand $(r\text{-}2)^3$.

Hint for b); The eigenvectors are identical, transform the system into one difference equation.

## 3. Optimal control (30 points)

Consider an investment problem where the flow of profits is given by the concave production function *f*(*k*). The opportunity cost per unit of time of holding a capital stock *k*, is given by the rental cost minus the capital gain on capital, i.e., $p_t(d+r)k_t-\dot{p}_t k_t$, where $p_t$ is the price of capital, ***d*** is depreciation and *r* is the discount rate. The maximization problem is thus.

$$\max_{\{k_s\}_0^T}\int_0^T e^{-rt}\left(f(k_t)-\left(p_t(d+r)-\dot{p}_t\right)k_t\right)dt \qquad (1)$$

$$k_0=0$$

a) State the first order condition that (implicitly) determines the optimal capital stock at *t*. (Hint; is this a dynamic problem?)

Now assume that the capital stock cannot be adjusted freely. Instead, a non-negative adjustment cost has to be born anytime investments are non-zero. The adjustment cost per unit of time is a convex function of the investment rate, denoted *c*(*i*) with *c*(0)=0.

Assume for simplicity that the price of capital is constant. The maximization problem then becomes

$$\max_{\{i_s\}_0^T} \int_0^T e^{-rt}\left( f(k_t) - i_t p - c(i_t) \right) dt$$
$$s.t. \qquad \dot{k}_t = i_t - dk_t \qquad (2)$$
$$k_0 = 0$$

b) Describe in words why the problem now cannot be solved in the same way as under a).

c) Write the current value Hamiltonian and the necessary conditions for an optimum.

Now parameterize as follows

$$f(k) = ak - \frac{bk^2}{2},$$
$$c(i) = \frac{gi^2}{2} \qquad (3)$$

with all parameters being strictly positive.

d) Express the solution to the dynamic optimality conditions as system of linear differential equations for $i_t$ and $k_t$. Is the system stable, explosive or saddle-path stable?

e) Draw a phase diagram of the system with *k* on the *x*-axis and *i* on the *y*-axis. Be careful when you draw the arrows.

f) Compare the optimal initial investment rate, $i_0$, for the case of finite and an infinite horizon *T*. Which is the larger one? (Hint; in the infinite horizon case, the solution settles down to a steady state. Use the phase diagram and draw the two optimal paths. Show why one has to be below the other.)

g) Which parameters determine the speed of convergence to the steady state? What is the relation between these parameters and the speed of convergence?

*Let the system be in a steady state. Assume that there is an unexpected fall marginal productivity, i.e., *b* increases, show what happens dynamically. (Hint: This can be done by drawing the phase diagram on top of he old one and setting the initial condition equal to the old steady state.)

## 4. Dynamic programming (20 points)

Consider the problem

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} b^t U(c_t)$$
$$s.t. \quad k_{t+1} = f(k_t) - c_t, \quad (4)$$
$$k_0 = k,$$
$$k_t \geq 0 \quad \forall t.$$

a) Substitute from the dynamic budget constraint to get rid of $c_t$. Write the Bellman equation as a maximum over $k_{t+1}$ with the first order condition for a maximum. Make sure all time subscripts are correct.

b) Use the envelope theorem to calculate $V'(k_t)$ and use the result to get rid of $V'(\cdot)$ in your answer to a).

Now let $U$ and $f$ be given by

$$U(c_t) = \ln c_t$$
$$f(k_t) = k_t^a \quad (5)$$
$$0 < a < 1.$$

c) Show that $V(k_t) = A \ln k_t + C$ solves the Bellman equation and solve for $A$ in terms of the parameters of the problem. You do not need to solve for $C$.

d) *Now assume there is habit persistence so the per period utility function is

$$U = U(c_t, c_{t-1}) \quad (6)$$

Write the Bellman equation with its associated first order condition. (Hint; the previous period's consumption level is now a state variable.)

## 5. Numerical Optimization (20 points)

Let **x** be a 2x1 vector with the elements $x_1$ and $x_2$. Suppose we want to find $\arg\max_{\mathbf{x}} f(\mathbf{x})$ by solving the following set of first order conditions with the Newton-Raphson method.

$$\begin{aligned} \frac{\partial f}{\partial x_1} &\equiv f_1(x_1, x_2) = 0, \\ \frac{\partial f}{\partial x_2} &\equiv f_2(x_1, x_2) = 0. \end{aligned} \tag{7}$$

a) Write down the equation that defines the Newton-Raphson algorithm for this problem.

b) Calculate the updated value $\mathbf{x}^1$ if we start at $\mathbf{x}^0 = \{0,0\}$ if

$$\begin{aligned} f_1(x_1, x_2) &= x_1 e^{x_2}, \\ f_2(x_1, x_2) &= x_2 e^{2x_1}. \end{aligned} \tag{8}$$