# 1. Introduction

The issue:

Choose from a set of admissible paths or functions $x(t) \in A$ the one that maximizes a given objective *functional* which associates a particular value to each admissible path $V[x(t)]$.

Example: resource (oil) extraction. Choose an extraction plan $x(t)$ stating how much is left in the well. To be admissible $x(t)$ has to satisfy:

$$\begin{aligned} &x(0) = K, \\ &\dot{x}(t) \geq 0, \forall t, \\ &x(T) \geq 0. \end{aligned} \tag{1.1}$$

The value (or objective) function associates a number to each path. This is given by

$$\begin{aligned} V[x(t)] &= \int_0^T e^{-rt} (p(\dot{x}(t)) - c(\dot{x}(t))dt \\ &= \int_0^T F(t, x(t), \dot{x}(t))dt. \end{aligned} \tag{1.2}$$

Three approaches in this course:

1. Calculus of Variation (Newton, Bernouilli)

$$\begin{aligned} &\max_{x(t)} \int_0^T F(t, x(t), \dot{x}(t))dt \\ &s.t.\ x(t) \text{ is admissible.} \end{aligned} \tag{1.3}$$

2. Optimal Control (Pontryagin)

$$
\begin{aligned}
&\max_{u(t)} \int_0^T F(t, x(t), u(t)) dt \\
&s.t. \\
&\dot{x}(t) = g(t, x(t), u(t)) \\
&x(0) = A, \\
&x(T) = Z, \\
&u(t) \text{ admissible for all } t.
\end{aligned} \tag{1.4}
$$

3. Dynamic Programming (Bellman).

$$
\begin{aligned}
&V(k_0) = \max \sum_{t=1}^{T} F(t, x_t, u_t) \\
&s.t. \\
&x_{t+1} = g(t, x_t, u_t) \\
&x_1 = A, \\
&x_T = Z.
\end{aligned} \tag{1.5}
$$

Example:

$$
\begin{aligned}
&\max_{c,k} \sum_{t=0}^{T} \beta^t U(c_t) \\
&s.t. \quad k_0 = \bar{k}_0, \\
&\qquad k_{t+1} = f(k_t) - c_t,\ t = 0, \ldots, T, \\
&\qquad k_{T+1} = 0
\end{aligned} \tag{1.6}
$$

Here we start from next to last period and solve the two period problem.

$$
V_{T-1} = \max V_{T-1}(k_{T-1}) = \max_{k_T} U(f(k_{T-1}) - k_T) + \beta U(f(k_T)). \tag{1.7}
$$

where we have substituted from terminal condition and transition equation. Note the $V_{T-1}(k_{T-1})$ is a function; it gives the value of what can be achieved during the remaining periods if the optimal plan is followed.

This certainly depends on how much capital one enters the current period with. Such variables are called *state variables*.

Given $V_{T-1}(k_{T-1})$ we can go one step further back and solve

$$V_{T-2}(k_{T-2}) = \max_{k_{T-1}} U(f(k_{T-2}) - k_{T-1}) + \beta V(k_{T-2})). \tag{1.8}$$

going back to the start point we have the whole solution. All these methods can handle (some) infinite horizon problems.

# 2. Dynamic Optimization in Continuous Time

## 2.1 Calculus of Variation

Look at the following fundamental dynamic problem

$$\begin{aligned}&\max_x \int_0^T F(t, x(t), \dot{x}(t))dt \\ &s.t. \quad x(0) = x_0, \; x(T) = x_T\end{aligned} \tag{2.1}$$

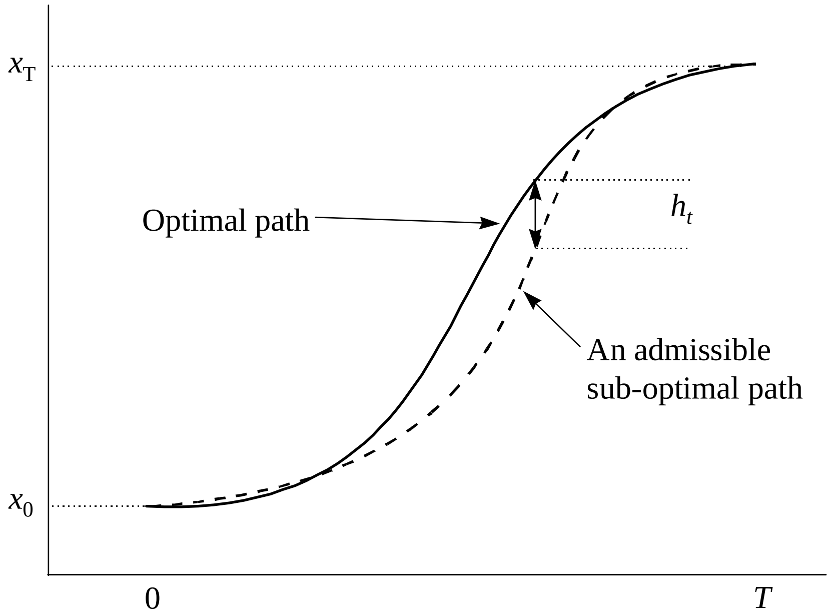
where $F$ is differentiable in its arguments. The problem is dynamic since $\dot{x}(t)$ is included. Otherwise we could maximize point by point in time.

An economic example could be that $F$ represents profits from a firm that make output by employing labor (x). Time enters the profit function since the firm discounts future profits. If changes in the number of persons employed is costly $\dot{x}(t)$ also enters the profit function. The firm can then not just in each moment hire the number of persons that maximize current profits.

A solution to the problem is a function $x^*(t)$ (with a continuous derivative). To find it we try to find some characteristics of it that can help us to search. We will in particular now derive some *necessary* conditions that the solution must satisfy. From them we *may* find the solution.

The trick is to define *admissible* deviations. These are the differences between the optimal path and an admissible but sub-optimal path. Let us fix a particular admissible path $x(t)$ and call the difference between this and the optimal path $h(t)$, i.e. $h(t) \equiv x^*(t) - x(t)$. We have now fixed a particular admissible path as in the figure below.

The constraints in (2.1) imply that $h(0) \equiv h(T) \equiv 0$ which in this case are the only admissibility constraints (together with differentiability).




Now look at a linear combination of the optimal path and an admissible deviation. For any constant *a* let

$$y(a,t) = x^*(t) + ah(t) \tag{2.2}$$

which is clearly admissible. Note that *y*(*a*,.) defines a particular path so for each value of *a* we have a particular path. We say that *y*(*a*,.) is a *one parameter family of admissible paths.* Since we can calculate a value of the objective for each path, this means that for each value of *a* we can associate a particular value of the objective. Now define the value of the program if we use *y*(*a*,.)

$$J(a) \equiv \int_0^T F\left( t, \overbrace{x^*(t) + ah(t)}^{y(a,t)}, \overbrace{\dot{x}^*(t) + a\dot{h}(t)}^{\dot{y}(a,t)} \right) dt \tag{2.3}$$

*J*(*a*) is thus an ordinary *function* of *a,* not a functional. We can then use standard optimisation techniques. This value must by assumption be maximized when *a*=0. We also find a standard necessary first order condition for a maximum

$$J'(0) = \frac{\partial}{\partial a}\left(\int_0^T F\left(t, \overbrace{x^*(t) + ah(t)}^{y(a,t)}, \overbrace{\dot{x}^*(t) + a\dot{h}(t)}^{\dot{y}(a,t)}\right)dt\right)\Bigg|_{a=0} = 0 \qquad (2.4)$$

Denote the three first derivatives of $F$ by $F_t, F_x, F_{\dot{x}}$ and suppress arguments when unnecessary. We then have

$$\Rightarrow 0 = \left(\int_0^T \frac{\partial}{\partial a} F(t, y(a,t), \dot{y}(a,t))dt\right)\Bigg|_{a=0}$$

$$= \int_0^T \left(F_x \frac{\partial y(a,t)}{\partial a} + F_{\dot{x}} \frac{\partial \dot{y}(a,t)}{\partial a}\right)dt \qquad (2.5)$$

$$= \int_0^T \left(F_x h(t) + F_{\dot{x}}\dot{h}(t)\right)dt$$

Let the total time derivative $dF_{\dot{x}}/dt$ be denoted $\dot{F}_{\dot{x}}$. We can then (2.5)by integrating $F_{\dot{x}}\dot{h}$ by parts

$$\int_0^T \overbrace{F_{\dot{x}}}^{u}\overbrace{\dot{h}dt}^{dv} = \left[\overbrace{F_{\dot{x}}}^{u}\overbrace{h}^{v}\right]_0^T - \int_0^T \overbrace{h}^{v}\overbrace{\dot{F}_{\dot{x}}dt}^{du}$$

$$= F_{\dot{x}(T)}\overbrace{h(T)}^{=0} - F_{\dot{x}(0)}\overbrace{h(0)}^{=0} - \int_0^T h\dot{F}_{\dot{x}}dt. \qquad (2.6)$$

Substituting this into (2.5) we find that the necessary condition is that along the optimal path

$$\int_0^T \left(F_x - \dot{F}_{\dot{x}}\right)hdt = 0 \qquad (2.7)$$

But this must be true for all the infinitely many different admissible deviations *h*. This requires that the value within parenthesis in (2.7) is zero for all *t* within the planning horizon.

$$F_x\left(t, x^*, \dot{x}^*\right) = \frac{dF_{\dot{x}}\left(t, x^*, \dot{x}^*\right)}{dt}, \qquad (2.8)$$
$$\forall t \subseteq [0, T].$$

This is the *Euler Equation* for the problem. Note the RHS of (2.8) is the total time derivative. We can then write this as

$$\begin{aligned} F_x\left(t, x^*, \dot{x}^*\right) &= \frac{\partial F_{\dot{x}}\left(t, x^*, \dot{x}^*\right)}{\partial t} + \frac{\partial F_{\dot{x}}\left(t, x^*, \dot{x}^*\right)}{\partial x^*}\frac{dx^*(t)}{dt} + \frac{\partial F_{\dot{x}}\left(t, x^*, \dot{x}^*\right)}{\partial \dot{x}^*}\frac{d\dot{x}^*(t)}{dt} \\ &\equiv F_{t,\dot{x}}\left(t, x^*, \dot{x}^*\right) + F_{x,\dot{x}}\left(t, x^*, \dot{x}^*\right)\dot{x}^*(t) + F_{\dot{x},\dot{x}}\left(t, x^*, \dot{x}^*\right)\ddot{x}^*(t)\,. \end{aligned} \qquad (2.9)$$

The Euler equation is thus a non-linear second order differential equation for the optimal path. Sometimes we may be able to solve for the function $x^*(t)$. At least we can derive some properties of it.

Since the solution to a second order differential equation contains two arbitrary constants we need two more equations to pin down the solution. For this we use the terminal and start conditions in (2.1).

Below we will see that (2.8) can be interpreted as a an arbitrage condition between different points in time

### A Simple Consumption Example

$$\begin{aligned} &\max_{c_t} \int_0^T e^{-rt} U(c_t)\,dt \\ &s.t. \quad \dot{K}_t = iK_t + w_t - c_t \\ &\qquad\quad K_0 = k_0, \quad K_T = k_T. \end{aligned} \qquad (2.10)$$

$$\begin{aligned} F\left(t, K, \dot{K}\right) &= e^{-rt} U\left(\underbrace{iK_t + v_t - \dot{K}_t}_{c_t}\right), \\ F_K &= e^{-rt} i U'(c_t), \\ F_{\dot{K}} &= -e^{-rt} U'(c_t). \end{aligned} \qquad (2.11)$$

The Euler equation is

$$\begin{aligned} &\dot{F}_{\dot{K}} = F_K \\ &\Rightarrow -d\left(e^{-rt}U'(c_t)\right)\big/dt = ie^{-rt}U'(c_t). \end{aligned} \tag{2.12}$$

Note that we express the Euler equation in terms of $c$ rather than in $K$. This will make it easier to interpret and gives us a first order differential equation in $c$ instead of a second order in $k$. We could, of course, solve the problem in terms of $K$ instead.

Before solving we want to interpret the Euler equation by showing that it is an arbitrage condition between successive points in time. Integrate (2.12) from $t$ to $t+\Delta t$

$$\begin{aligned} &\Rightarrow -\int_t^{t+\Delta t} de^{-rt}(U'(c_t)) \\ &= e^{-rt}(U'(c_t)) - e^{-r(t+\Delta t)}(U'(c_{t+\Delta t})) \\ &= \int_t^{t+\Delta t} ie^{-rs}U'(c_t)ds. \end{aligned} \tag{2.13}$$

or

$$-U'(c_t) + e^{-r\Delta t}(U'(c_{t+\Delta t})) + \int_t^{t+\Delta t} ie^{-r(s-t)}U'(c_t)ds = 0 \tag{2.14}$$

Note that LHS of (2.14) is the gain in utility by saving one marginal unit of consumption at $t$, consuming the interest on the extra saving between $t$ and $t+\Delta t$ and consuming an extra unit of consumption at $t+\Delta t$. If the plan is optimal this should have zero effect on total utility,

Now we use (2.12) to try to find a solution

$$
\begin{aligned}
&-de^{-rt}(U'(c_t))\big/dt = r\cancel{e^{-rt}}(U'(c_t)) - \cancel{e^{-rt}}(U''(c_t))\dot{c}_t \\
&= i\cancel{e^{-rt}}U'(c_t) \qquad (2.15)\\
&\Rightarrow \dot{c}_t = \frac{U'(c_t)}{-U''(c_t)}(i-r)
\end{aligned}
$$

(2.15) tells us a lot of the optimal path although we may be unable to solve for the level of consumption.

Note that (2.15) must hold for the solution to be optimal also for a non-constant interest rate. This is intuitive in light of (2.14).

By specifying a utility function we can go further.

In the CARA utility (exponential) case

$$
U = \frac{-e^{-\lambda c}}{\lambda}, \qquad U' = e^{-\lambda c}, \qquad U'' = -\lambda e^{-\lambda c} \qquad (2.16)
$$

so

$$
\dot{c}_t = \frac{i-r}{\lambda}, \;\; c_t = c_0 + \frac{i-r}{\lambda}t. \qquad (2.17)
$$

Note that this just defines the slope of the optimal path, the level is determined from the dynamic budget constraint.

$$
\dot{K}_t - iK_t = w_t - c_t = w_t - c_0 - \frac{i-r}{\lambda}t \qquad (2.18)
$$

Multiplying by the integration factor and integrating we have

$$
\begin{aligned}
&e^{-it}\left(\dot{K}_t - iK_t\right) = \frac{de^{-it}K_t}{dt} = w_t - c_0 - \frac{i-r}{\lambda}t \\
&e^{-iT}k_T - k_0 = \int_0^T e^{-it}w_t dt - \int_0^T e^{-it}c_t dt.
\end{aligned}
\qquad (2.19)
$$

Now define

$$\int_0^T e^{-it} v_t dt + k_0 - e^{-iT} k_T \equiv W_0 = \int_0^T e^{-it} c_t dt. \tag{2.20}$$

This is the *intertemporal* budget constraint. Solving this

$$\begin{aligned}
&\int_0^T e^{-it}\left(c_0 + \frac{i-r}{\lambda}t\right)dt = -\left[c_0 \frac{e^{-it}}{i}\right]_0^T + \int_0^T \overbrace{\left(\frac{i-r}{\lambda}t\right)}^{u} \overbrace{e^{-it}dt}^{dv} \\
&= -\left[c_0 \frac{e^{-it}}{i}\right]_0^T - \left[\frac{i-r}{\lambda} t \frac{e^{-it}}{i}\right]_0^T - \int_0^T -\frac{i-r}{\lambda}\frac{e^{-it}}{i} dt \\
&= -\left[c_0 \frac{e^{-it}}{i}\right]_0^T - \left[\frac{i-r}{\lambda} t \frac{e^{-it}}{i}\right]_0^T - \left[\frac{i-r}{\lambda}\frac{e^{-it}}{i^2}\right]_0^T \\
&= c_0 \left(\frac{1-e^{-iT}}{i}\right) + \frac{i-r}{\lambda i^2} - e^{-iT}\left(\frac{i-r}{\lambda i}T + \frac{i-r}{\lambda i^2}\right) = W_0.
\end{aligned} \tag{2.21}$$

Note that if $i=r$ consumption is simply a fraction of wealth, that decreases with the length of the planning horizon.

$$c_0 = \frac{i}{(1-e^{-iT})} W_0 \tag{2.22}$$

So with an infinite horizon $c_t = iW_t$.

Similarly on the case of CRRA utility

$$U = \frac{c^{1-1/\sigma}}{1-1/\sigma}, \qquad U' = c^{-1/\sigma}, U'' = -\frac{c^{-1/\sigma-1}}{\sigma} \tag{2.23}$$

the Euler equation (2.15) becomes

$$\begin{aligned}
&\dot{c}_t / c_t = \sigma(i-r) \\
&\Rightarrow c_t = c_0 e^{\sigma(i-r)t}.
\end{aligned} \tag{2.24}$$

Using the intertemporal budget constraint

$$\int_0^T e^{-it} c_0 e^{\sigma(i-r)t} dt = \int_0^T c_0 e^{((\sigma-1)i-\sigma r)t} dt = W_0$$
$$\Rightarrow c_0 = \left( \frac{(\sigma r - (\sigma - 1)i)}{1 - e^{((\sigma-1)i-\sigma r)T}} \right) W_0 \tag{2.25}$$

Note the results when σ=1 and when $T \to \infty$.

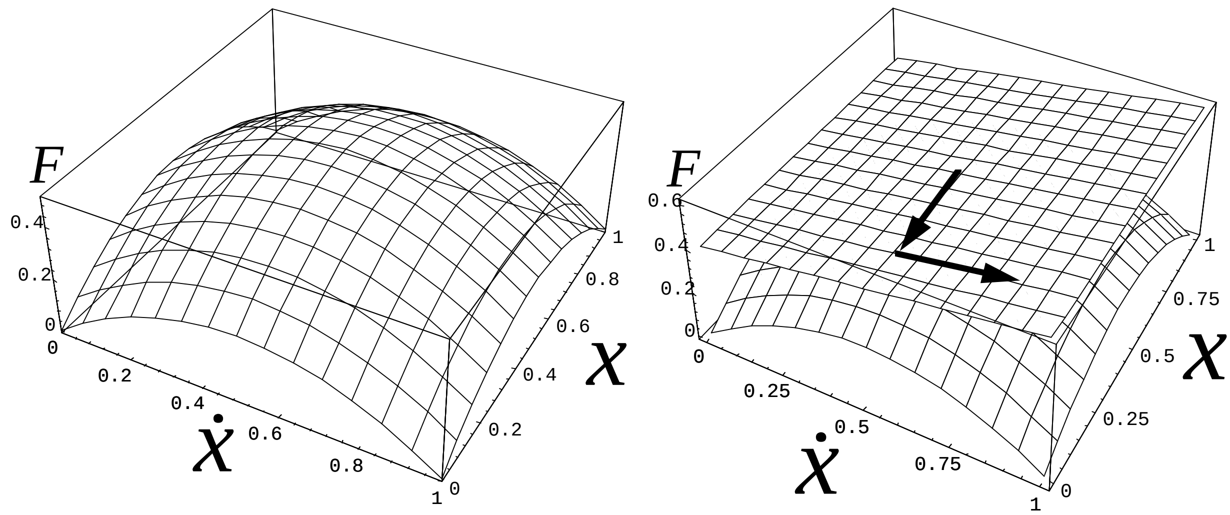
**A Sufficient Condition**

The Euler condition is necessary but not sufficient. It is however also sufficient for a maximum if $F(t,x,\dot{x})$ is concave in $x,\dot{x}$ (still assuming a fixed finite time horizon).

Recall that if $F(t,x,\dot{x})$ is concave in $x,\dot{x}$ then

$$F(t,x,\dot{x})$$
$$\leq F\left(t,x^*,\dot{x}^*\right) + \left(x - x^*\right) F_x\left(t,x^*,\dot{x}^*\right) + \left(\dot{x} - \dot{x}^*\right) F_{\dot{x}}\left(t,x^*,\dot{x}^*\right). \tag{2.26}$$

To see this




Assume that $F\left(t,x_t^*,\dot{x}_t^*\right) \equiv F^*$ satisfies the Euler equation and $F$ is concave in $x,\dot{x}$. We then want to show that $F\left(t,x_t^*,\dot{x}_t^*\right)$ is optimal, i.e., that

$$\int_0^T F\left(t, x_t, \dot{x}_t\right)dt \le \int_0^T F\left(t, x_t^*, \dot{x}_t^*\right)dt \tag{2.27}$$

for all admissible paths. Admissible deviations are defined $h(t) \equiv x(t) - x^*(t)$ with $\dot{h}(t) \equiv \dot{x}(t) - \dot{x}^*(t)$.

Now using (2.26) we have that

$$\begin{aligned} &F \le F^* + \overbrace{(x - x^*)}^{h} F_x^* + \overbrace{(\dot{x} - \dot{x}^*)}^{\dot{h}} F_{\dot{x}}^* \\ &\Rightarrow \int_0^T F dt \le \int_0^T F^* dt + \int_0^T \left(hF_x^* + \dot{h}F_{\dot{x}}^*\right)dt \end{aligned} \tag{2.28}$$

By integrating by parts we find that

$$\begin{aligned} \int_0^T \left(hF_x^* + \dot{h}F_{\dot{x}}^*\right)dt &= \int_0^T \left(hF_x^*\right)dt + \left[F_{\dot{x}}^* h\right]_0^T - \int_0^T h\dot{F}_{\dot{x}}^* dt \\ &\qquad F_{\dot{x}_T} \overbrace{h_T}^{=0} - F_{x_0} \overbrace{h_0}^{=0} + \int_0^T h\left(F_x^* - \dot{F}_{\dot{x}}^*\right)dt \\ &= 0. \end{aligned} \tag{2.29}$$

This shows that concavity, which gave us the inequality in (2.28), makes the Euler equation sufficient for optimality. Note that we require global concavity.

## Transversality Conditions

Assume now that $k_{T_1}$ is free. Before we used the terminal condition for $k_T$ to find one integration constant. Now we need some other condition to do this – the *transversality condition.*

An admissible deviation $h$ is now *not* required to satisfy $h(T)$=0. The necessary condition (2.4), (2.5) are still valid but (2.6) is changed slightly.

$$
\begin{aligned}
\int_0^T F_{\dot{x}} \dot{h} dt &= \left[ F_{\dot{x}} h \right]_0^T - \int_0^T h \dot{F}_{\dot{x}} dt \\
&= F_{\dot{x}(T)} \overbrace{h(T)}^{\neq 0} - F_{x(0)} \overbrace{h(0)}^{=0} - \int_0^T h \dot{F}_{\dot{x}} dt.
\end{aligned} \tag{2.30}
$$

So the necessary condition becomes that along the optimal path

$$
\int_0^T \left( F_x - \dot{F}_{\dot{x}} \right) h dt + F_{\dot{x}} \left( T, x^*(T), \dot{x}^*(T) \right) h(T) = 0. \tag{2.31}
$$

So we see that the Euler equation is still valid. In addition to the Euler equation we have the added condition

$$
F_{\dot{x}} \left( T, x^*(T), \dot{x}^*(T) \right) \tag{2.32}
$$

This is the transversality condition when the horizon is fixed but the endpoint is free.

### Example

In the consumption example (2.10) we expect that if no end condition is set for $K$ it must be optimal to consume so much that marginal utility goes to zero. Let's verify that.

$$
F_{\dot{K}}(t_1, K_{t_1}, \dot{K}_{t_1}) = -e^{-rt_1} U'(c_{t_1}). \tag{2.33}
$$

### More General Transversality Conditions

Now look at the case when both the endpoint and the horizon is free. We still use $a$ to index admissible deviations as in (2.2). Now we also have to consider admissible deviations in the terminal time $T$. Thus we let the admissible deviation from the optimal terminal time $T^*$ be an arbitrary number $\Delta T$ so that the terminal time associated with the path $y(a,t)$ is $T(a)=a\Delta T$ so $\partial T(a)/\partial a = \Delta T$.

The derivative of the whole program can now be written

$$J'(a) = \frac{\partial}{\partial a}\left(\int_0^{T(a)} F\left(t, \overbrace{x^*(t) + ah(t)}^{y(a,t)}, \overbrace{\dot{x}^*(t) + a\dot{h}(t)}^{\dot{y}(a,t)}\right)dt\right)\Bigg|_{a=0}$$

$$= \int_0^{T(a)} \left(F_x h(t) + F_{\dot{x}}\dot{h}(t)\right)dt + F(T, y(a,t), \dot{y}(a,t))\frac{\partial T(a)}{\partial a}$$

$$= \int_0^{T(a)} h(t)\left(F_x - \dot{F}_{\dot{x}}\right)dt + F_{\dot{x}(T)}h(T) + F(T, y(a,t), \dot{y}(a,t))\Delta T.$$

where we used the integration by parts as above in the third equality. We want to express the term $h(T)$ in terms of variations in the final value of the state variable $x$ denoted $\Delta x$ and variations in the stopping time $\Delta T$. The terminal value of the state variable can be approximated $\Delta x = h(T^*) + \left(\dot{y}(T^*)\right)\Delta T$ which implies that $h(T^*) = \Delta x - \left(\dot{y}(T^*)\right)\Delta T$. Substituting this into the first order condition we get.

$$\begin{aligned} &0 \\ &= \int_0^{T(a)} h(t)\left(F_x - \dot{F}_{\dot{x}}\right)dt + F_{\dot{x}(T)}(\Delta x - \dot{y}(T)\Delta T) + F(T, y(a,t), \dot{y}(a,t))\Delta T \quad (2.34) \\ &= \int_0^{T} h(t)\left(F_x - \dot{F}_{\dot{x}}\right)dt + F_{\dot{x}(T)}\Delta x + \left[F(T, x(T), \dot{x}(T)) - F_{\dot{x}(T)}\dot{x}(T)\right]\Delta T \end{aligned}$$

If $\Delta x$ and $\Delta T$ are free we require

$$\begin{aligned} &F_{\dot{x}(T)} = 0, \\ &\left[F(T, x(T), \dot{x}(T)) - F_{\dot{x}(T)}\dot{x}(T)\right] = 0 \end{aligned} \quad (2.35)$$

to have (2.34) satisfied. Note that the stars are suppressed in the last equation.

## More on the Interpretation of the Euler Equation

Now let us integrate the Euler equation (2.8) from 0 to *T*.

$$\begin{aligned}\int_0^T F_x\left(t, x^*, \dot{x}^*\right) dt &= \int_0^T \frac{dF_{\dot{x}}\left(t, x^*, \dot{x}^*\right)}{dt} dt \\ &= F_{\dot{x}}\left(T, x^*, \dot{x}^*\right) - F_{\dot{x}}\left(0, x^*, \dot{x}^*\right).\end{aligned} \tag{2.36}$$

When the endpoint is free we know that $F_{\dot{x}}\left(T, x^*, \dot{x}^*\right)$ is zero. From (2.36) we then see that $-F_{\dot{x}}\left(0, x^*, \dot{x}^*\right)$ equals the sum of marginal benefits of an extra "capital" unit over the full horizon. Alluding to the envelope theorem we then understand that $-F_{\dot{x}}\left(0, x^*, \dot{x}^*\right)$ is the shadow value of an extra unit of the state variable at time zero. (see the example in (2.33) This turns out to be true also for all other points in time. But note that these value are as seen from time 0 (if there is discounting this matters). The transversality conditions in (2.35) are then easy to interpret. If the terminal value of the state variable is free, its shadow value should be driven to zero. If the stopping time is free, the *sum* of current profits and the generation of future profits should be driven to zero.

## Infinite Horizon

The intuition for the Euler equation above as an arbitrage between successive time points still suggests that it is valid also in infinite horizon problems. This is the case for properly specified economic problems where the objective function converge to something finite for all admissible deviations. If this is not the case optimality becomes ambiguous.

To find the optimal path we have to modify the optimality conditions. We must now make sure that $F_{\dot{x}(T)}\Delta x + \left[F(T, x(T), \dot{x}(T)) - F_{\dot{x}(T)}\dot{x}(T)\right]\Delta T$ goes to zero as the horizon *T* goes to infinity. For this purpose we require

$$\lim_{T\to\infty} F_{\dot{x}(T)} = 0,$$
$$\lim_{T\to\infty}\left[F(T,x(T),\dot{x}(T)) - F_{\dot{x}(T)}\dot{x}(T)\right] = 0 \tag{2.37}$$

So the shadow value should go to zero when *x* is free also in the limit. The sum of current and future profits should also go to zero since in an infinite horizon problem, there is no fixed stopping time.

In economic infinite horizon models we often want to find a steady state solution $x^{SS}$ (s.t. $\dot{x} = \ddot{x} = 0$) for some properly detrended variable. This works if time does not enter *F* or just as an exponential discounting. The problem is then (time) autonomous. If we take this steady state to be the boundary condition

$$\lim_{t\to\infty} x_t = x^{ss} \tag{2.38}$$

or

$$\lim_{t\to\infty} \dot{x}_t = 0 \tag{2.39}$$

we have the necessary information to find the solution with integration constants. Take the Ramsey problem as an example.

$$\max_k \int_{t_0}^{\infty} e^{-\theta t} U(c_t)\,dt$$
$$s.t. \quad \dot{k}_t = f(k_t) - c_t, \tag{2.40}$$

and the initial and boundary conditions

$$k_0 = \bar{k}$$
$$\lim_{t\to\infty} \dot{k}_t = 0. \tag{41}$$

We then have

$$
\begin{aligned}
&F(t,k,\dot{k}) = e^{-\theta t}U\left(f(k)-\dot{k}\right) \\
&F_k = e^{-\theta t}U'f'(k) \\
&F_{\dot{k}} = -e^{-\theta t}U'\left(f(k)-\dot{k}\right) = -e^{-\theta t}U'(c) \\
&\frac{dF_{\dot{k}}}{dt} = \theta e^{-\theta t}U' - e^{-\theta t}\frac{dU'}{dt} = \theta e^{-\theta t}U' - e^{-\theta t}U''\dot{c}
\end{aligned}
\tag{2.42}
$$

Note that in the following I find expressions for $c$ rather than for $k$. Note also that the discounting terms cancels. So the Euler equation can be written

$$
\begin{aligned}
&U'f'(k) = \theta U' - U''\dot{c} \\
&\Rightarrow \dot{c} = \frac{U'}{-U''}\left(f'(k)-\theta\right)
\end{aligned}
\tag{2.43}
$$

This gives us the system of differential equations

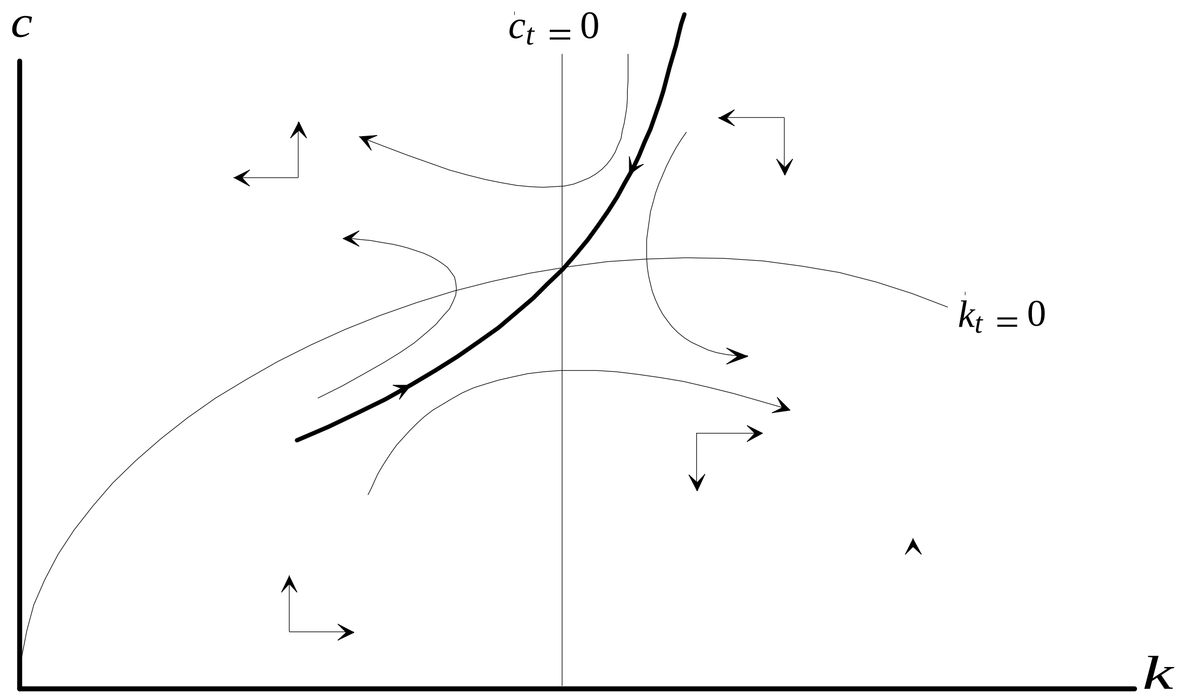
$$
\begin{aligned}
\dot{c}_t &= \frac{U'}{-U''}\left(f'(k_t)-\theta\right) \\
\dot{k}_t &= f(k_t) - c_t.
\end{aligned}
\tag{2.44}
$$

Note that in terms of $k$ the Euler equation is

$$
\begin{aligned}
&\cancel{e^{-\theta t}}U'f'(k) = \theta\cancel{e^{-\theta t}}U' - \cancel{e^{-\theta t}}U''f'\dot{k} + \cancel{e^{-\theta t}}U'\ddot{k} \\
&U'f'(k) = \theta U' - U''f'\dot{k} + U'\ddot{k},
\end{aligned}
\tag{2.45}
$$

i.e., a second order differential equation. Note the equivalence between a second order differential equations and a two equation system of differential equations.

The system (2.44) has variable coefficients and may be difficult to solve analytically. But we have already seen that it can be analyzed qualitatively. First we clearly have steady state. Secondly we can draw its phase diagram.




Note that as always all paths (arrows) in the figure satisfy the Euler equation. The initial and boundary conditions together pick just one path. Only the saddle path satisfies the boundary condition and by knowledge of $k_0$ we can then find $c_0$. We have thus pinned down just one path and (implicitly) found the two integration constants.