2.2 Optimal Control

In an optimal control problem we make a distinction between control variables (e.g., consumption or investments) and state variables (e.g., capital stocks or debt) that are governed by a differential equation (*transition* equation) and thus given in each point in time.

$$\max_u \int_{t_0}^{t_1} f(t, x(t), u(t))dt \tag{2.46}$$
$$s.t. \quad \dot{x}(t) = g(t, x(t), u(t)), \qquad x(0) = x_0.$$

This is the simplest problem in optimal control, note that it has no terminal condition for the state variable. As a means to finding a solution we define a multiplier function λ(*t*) for the transition equation.

For a combination of *x, u* to be admissible it must be that ∀ $t \subseteq [t_0, t_1]$ $g(t, x(t), u(t)) - \dot{x}(t) = 0$. Adding zero to the instantaneous payoff function each point in time will not change the problem. The following rewriting of the problem will prove to be useful

$$\int_{t_0}^{t_1} f(t, x(t), u(t)) + \lambda(t)\big(g(t, x(t), u(t)) - \dot{x}(t)\big)dt \tag{2.47}$$

Now we want to get rid $\dot{x}(t)$. So integrate by parts

$$\int_{t_0}^{t_1} \lambda\dot{x}dt = [\lambda x]_{t_0}^{t_1} - \int_{t_0}^{t_1} \dot{\lambda}xdt \tag{2.48}$$

giving

$$\int_{t_0}^{t_1} \left(f(t,x,u) + \lambda g(t,x,u) + \dot{\lambda} x \right) dt - \lambda(t_1) x(t_1) + \lambda(t_0) x(t_0) \quad (2.49)$$

Now we use the same procedure as when deriving necessary conditions for the Calculus of Variation problem. Instead of looking at admissible variations of x we look at admissible variations of the control variable u. Let u^* represent the optimal control and u some other admissible control and define $h= u^*-u$ Let $y(a,t)$ denote the state variable generated by using the control u^*+ah. Let $J(a)$ denote the value of the program (2.49) when using the control u^*+ah. Clearly $J(0)$ is the maximum of J by definition and $J'(0)=0$. As before we will use this necessary condition to find necessary properties of the solution.

$$\begin{aligned} J(a) \equiv & \int_{t_0}^{t_1} \left(f\left(t, y(a,t),\, u^*(t) + ah(t)\right) + \lambda g\left(t, y(a,t),\, u^*(t) + ah(t)\right) + \dot{\lambda}(t) y(a,t) \right) dt \\ & -\lambda_{t_1} y(a,t_1) + \lambda_{t_0} y(a,t_0), \end{aligned} \quad (2.50)$$

$$\begin{aligned} J'(0) = & \int_{t_0}^{t_1} \left(f_x y_a(0,t) + f_u h + \lambda g_x y_a(0,t) + \lambda g_u h + \dot{\lambda} y_a(0,t) \right) dt \\ & -\lambda(t_1) y_a(0,t_1) + \lambda(t_0) y_a(0,t_0) \\ = & \int_{t_0}^{t_1} \left(\left(f_x + \lambda g_x + \dot{\lambda} \right) y_a(0,t) + \left(f_u + \lambda g_u \right) h \right) dt \\ & -\lambda(t_1) y_a(0,t_1) + \lambda(t_0) \overbrace{y_a(0,t_1)}^{=0}, \\ = & \, 0. \end{aligned} \quad (2.51)$$

So far we have not put any restrictions on λ. Now let it follow the differential equation

$$\begin{aligned} \dot{\lambda} &= -f_x - \lambda g_x \\ \lambda(t_1) &= 0. \end{aligned} \quad (2.52)$$

This takes care of the term multiplying $y_a(0,t)$ and produces a transversality condition for the terminal point in time.

For (2.51) and (2.52) to hold we also require that along the optimal path

$$\int_{t_0}^{t_1} (f_u + \lambda g_u) h dt = 0, \tag{2.53}$$

for all admissible deviations *h*. For this to hold for all such deviations we need that

$$f_u + \lambda g_u = 0, \quad \forall t \subseteq [t_0, t_1]. \tag{2.54}$$

We have now found that if we define λ(*t*) according to (2.52) the necessary condition for optimality can be written as (2.54). To remember this and the we construct the *Hamiltonian*.

$$H(t, x(t), u(t), \lambda(t)) = f(t, x(t), u(t)) + \lambda(t) g(t, x(t), u(t)), \tag{2.55}$$

from which we can derive the necessary conditions for optimality.

$$\begin{aligned} H_u &= f_u\left(t, x, u^*\right) + \lambda g_u\left(t, x_t, u_t^*\right) = 0 \\ -H_x &= -f_x\left(t, x, u^*\right) - \lambda g_x\left(t, x, u^*\right) = \dot{\lambda} \\ H_\lambda &= g_x\left(t, x, u^*\right) = \dot{x} \end{aligned} \tag{2.56}$$

and the initial condition $x(t_0) = x_0$ and the terminal condition $\lambda(t_1) = 0$.

Interpretations

To see the direct analogy of the Optimal Control approach and standard Lagrange approach look at a discrete time version of (2.46)

$$\begin{aligned}
&\max_{u} \sum_{t=t_0}^{t_1} f(t, x_t, u_t) \\
&s.t. \quad x_{t+1} - x_t = g(t, x_t, u_t), \\
&\qquad\quad x_{t_0} = x_0,\ x_{t_1} = x_1
\end{aligned} \tag{2.57}$$

The Lagrangean is

$$\begin{aligned}
L = &\sum_{t=t_0}^{t_1} f(t, x_t, u_t) + \lambda_t \left(g(t, x_t, u_t) - x_{t+1} + x_t \right) \\
&+ \mu_0 \left(x_0 - x_{t_0} \right) + \mu_1 \left(x_1 - x_{t_1} \right)
\end{aligned} \tag{2.58}$$

First order conditions are

$$f_u(t, x_t, u_t) + \lambda_t g_u(t, x_t, u_t) = 0 \tag{2.59}$$

This is directly analogous to the optimality condition(2.54).

Differentiate with respect to x_t to get

$$f_x(t, x_t, u_t) + \lambda_t \left(g_x(t, x_t, u_t) + 1 \right) - \lambda_{t-1} \tag{2.60}$$

We know that the interpretation of λ_t in this problem is the shadow value of capital. Not consider the following situation. Two individuals solve identical problems. At t-1 one decides to buy one marginal unit of capital from the other and to sell it back at t. If prices of capital reflect shadow values they must be λ_t and λ_{t-1}. The consequences on the total value of the program is then given by (2.60) where the first part is the direct increase in value, the second the resell value and the third the price to pay. We conclude that (2.60) =0. Rewriting we see the direct analogy to (2.52).

$$\begin{aligned}
&-f_x(t, x_t, u_t) - \lambda_t g_x(t, x_t, u_t) = \lambda_t - \lambda_{t-1} \\
&-f_x(t, x(t), u(t)) - \lambda(t) g_x(t, x(t), u(t)) = \dot{\lambda}(t)
\end{aligned} \tag{2.61}$$

(2.52) is thus a consequence of a correct valuation of x_t (not an optimality condition). The interpretation of $\lambda(t)$ is identical in the two problems.

Define the optimal value function to problem (2.46) as the value of the objective function using the optimal control. This obviously depends on the initial condition and the starting point.

$$V(x(t),t)=\int_t^{t_1} f\left(s,x^*,u^*\right)ds. \tag{2.62}$$

$$V_x(x,t)=\lambda(t). \tag{2.63}$$

This is also true at later points in time. Note that λ_{t+s} in (2.62) is the shadow value of the state variable at $t+s$ *seen from the time of start of the program*. We could also see this immediately from (2.49).

This implies that we can view the Hamiltonian $f(t,x(t),u(t))+\lambda(t)g(t,x(t),u(t))$ as representing the sum of the current profit (the first term) and the value of capital accumulation (future profits generated from accumulating the state variable). That we should maximize the Hamiltonian in each point in time is then an intuitive condition.

Sufficiency

As in the Calculus of Variation we get a sufficiency condition by imposing the right concavity condition. Assume that f and g are concave in x,u and $\lambda \geq 0$. This implies that the Hamiltonian is concave in x,u. Then since f is concave

$$\begin{aligned} &f \leq f^* + (x-x^*)\, f_x^* + (u-u^*)\, f_u^* \\ &\int_{t_0}^{t_1} f dt \leq \int_{t_0}^{t_1} f^* dt + \int_{t_0}^{t_1} \left((x-x^*)\, f_x^* + (u-u)\, f_u^*\right)dt. \end{aligned} \tag{2.64}$$

So we want to show that the last integral in (2.64) is ≤ 0. Substitute for f_x from (2.56) and then integrate by parts the term involving $\dot{\lambda}$ to get rid of that.

$$\int_{t_0}^{t_1}\left((x-x^*)\,f_x^* + (u-u^*)\,f_u^*\right)dt$$

$$= \int_{t_0}^{t_1}\left((x-x^*)\left(-\dot{\lambda}-\lambda g_x^*\right)+(u-u^*)(-\lambda g_u^*)\right)dt$$

$$= -\overbrace{\left[\lambda(x-x^*)\right]_{t_0}^{t_1}}^{0} + \int_{t_0}^{t_1}\overbrace{(\dot{x}-\dot{x}^*)}^{g-g^*}\lambda dt$$

$$+\int_{t_0}^{t_1}\left((x-x^*)\left(-\lambda g_x^*\right)+(u-u^*)(-\lambda g_u^*)\right)dt$$

$$= \int_{t_0}^{t_1}\lambda\left((g-g^*)-\left((x-x^*)g_x^*+(u-u^*)g_u^*\right)\right)dt \leq 0. \tag{2.65}$$

If $\lambda \leq 0$ we need that g is convex in x,u. Then H is still concave. If g is linear we see that its sufficient that f is concave in x,u.

Current Value Hamiltonian

Often we have problems where t only enters as an exponential discounting. E.g.,

$$\begin{aligned}&\max_u \int_0^T e^{-rt} f\left(x(t),u(t)\right)dt\\ &s.t. \quad \dot{x}(t) = g\left(x(t),u(t)\right), \quad x_{t_0} = x_0.\end{aligned} \tag{2.66}$$

The Hamiltonian with necessary conditions is

$$
\begin{aligned}
&H(t,x,u) = e^{-rt} f(x,u) + \lambda g(x,u) \\
&H_u = e^{-rt} f_u + \lambda g_u = 0 \\
&H_x = e^{-rt} f_x + \lambda g_x = -\dot{\lambda},\ \ \lambda_T = 0.
\end{aligned}
\tag{2.67}
$$

It is often convenient to use a *current* shadow value defined as

$$
\begin{aligned}
&e^{-rt}\mu(t) \equiv \lambda(t) \\
&\Rightarrow \dot{\lambda} = -re^{-rt}\mu + e^{-rt}\dot{\mu} = e^{-rt}(\dot{\mu} - r\mu)
\end{aligned}
\tag{2.68}
$$

Substitute into (2.67)

$$
\begin{aligned}
&H(t,x,u) = e^{-rt} f(x,u) + e^{-rt}\mu g(x,u) \\
&H_u = \cancel{e^{-rt}}(f_u + \mu g_u) = 0 \\
&H_x = \cancel{e^{-rt}}(f_x + \mu g_x) = -\cancel{e^{-rt}}(\dot{\mu} - r\mu),\ \ e^{-rT}\mu_T = 0.
\end{aligned}
\tag{2.69}
$$

We get rid of all the discounting factors by defining the *current value Hamiltonian* with associated necessary conditions.

$$
\begin{aligned}
&G(t,x,u) \equiv e^{rt}H(t,x,u) = f(x,u) + g\mu(x,u) \\
&G_u = f_u + \mu g_u = 0 \\
&G_x = f_x + \mu g_x = -(\dot{\mu} - r\mu),\ \ e^{-rT}\mu_T = 0.
\end{aligned}
\tag{2.70}
$$

If T is infinity we may not divide the transversality condition by the discount factor since it is zero.

Some infinite horizon results

The optimality condition and the differential equation for the shadow value and the state variable

$$
\begin{aligned}
H_u &= f_u\left(t, x_t, u_t^*\right) + \lambda_t g_u\left(t, x_t, u_t^*\right) = 0 \\
-H_x &= -f_x\left(t, x_t, u_t^*\right) - \lambda_t g_x\left(t, x_t, u_t^*\right) = \dot{\lambda}_t \\
H_\lambda &= g_x\left(t, x_t, u_t^*\right) = \dot{x}_t
\end{aligned}
\tag{2.71}
$$

are necessary also in the infinite horizon case. The transversality condition is the problem. If we are ready to assume that the optimal path eventually settles down to a steady state

$$\lim_{t\to\infty} x_t = x^s \tag{2.72}$$

we have enough information to solve for the optimal path. The phase diagram is one way to do this.

This would be the case in the Ramsey problem.

Another simple case if $H(t,x,u,\lambda)$ is concave in x,u. Then (2.71) together with the transversality condition

$$\lim_{t\to\infty} \lambda_t (x_t - x_t^*) \geq 0 \tag{2.73}$$

for admissible paths x provide *sufficient* conditions for optimality (Mangasarin). It seems economically reasonable to assume that $\lim_{t\to\infty} \lambda_t x_t^* = 0$, i.e. that the value seen from today of the capital stock infinitely far away into the future is zero. Then its enough to assume that the shadow value of capital and all admissible levels of x are positive for (2.73) to hold.

When the "terminal" state is free the transversality condition is "typically" the intuitive

$$\lim_{T\to\infty} \lambda(T) = 0 \tag{2.74}$$

or

$$\lim_{T\to\infty} e^{-rT} \mu_T . \tag{2.75}$$

In the free ending time problem and the infinite horizon case we also need the transversality condition that

$$\lim_{T\to\infty} H = 0 \tag{2.76}$$

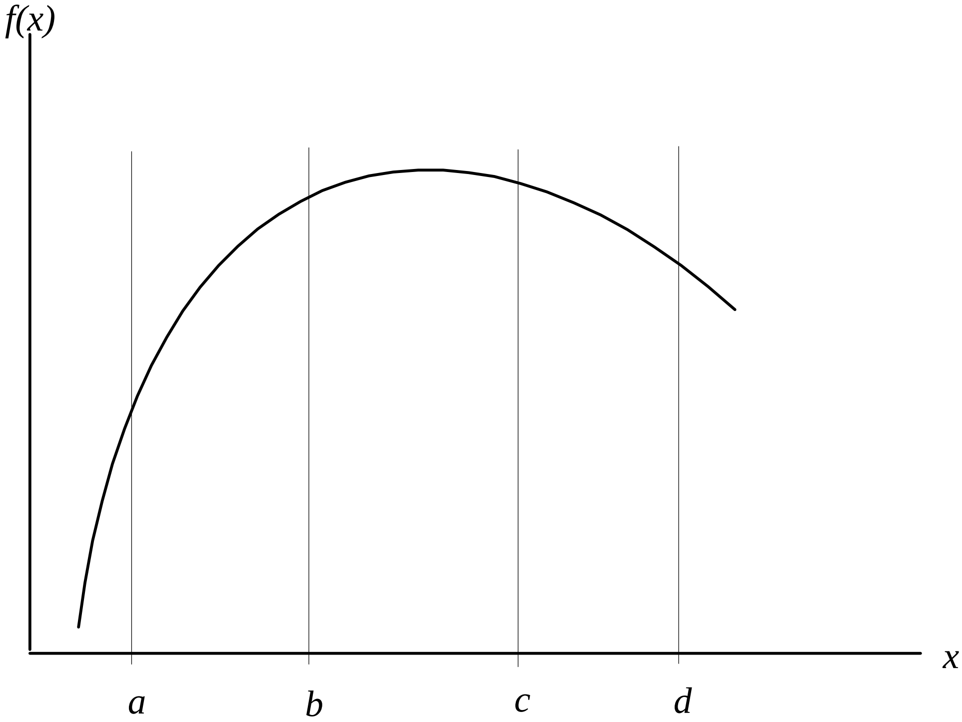
Profits (current and future) are not generated at the optimal stopping time or in infinity.

Bounded Controls

For a control to be optimal it is necessary that it solves

$$\max_{u} H(t, x_t, u_t, \lambda_t) \tag{2.77}$$

If u is bounded $H_u = 0$ is not necessary for an optimum. As in standard maximization the first order conditions only holds for interior solutions.



If we maximize $f(x)$ over $[a,b]$ b is optimal and $f'(x^*)$>0. If the range is $[c,d]$ c is optimal with $f'(x^*)$<0. Also in optimal control we may use *Kuhn-Tucker* multipliers in this case. Assume we solve problem (2.46) but restrict u to the range [a,b]. We then form the appended Hamiltonian

$$\begin{aligned} H(t, x_t, u_t, \lambda_t) = {} & f(t, x_t, u_t) + \lambda_t g(t, x_t, u_t) \\ & + w_1(b - u) + w_2(u - a). \end{aligned} \tag{2.78}$$

The optimality condition now becomes

$$
\begin{aligned}
&f_u + \lambda g_u - w_1 + w_2 = 0 \\
&w_1, w_2 \geq 0, \\
&w_1(b - u^*) = w_2(u^* - a) = 0.
\end{aligned}
\tag{2.79}
$$

Except for the knife-edge case we have that if w_1>0, $(b-u^*)$=0 so the H_u>0 as in the figure

The Pontryagin Maximum Principle

Now we come to a more general formulation of the necessary conditions for a solution to the optimal control problem. We allow for a any finite number of discontinuity points in the control, n control and state variables and that the controls are restricted to a constant weak subset of R^n.

$$
\begin{aligned}
&\max_{\mathbf{u}} \int_{\mathbf{0}}^{T} f(t,\mathbf{x},\mathbf{u})dt \\
&s.t. \quad \dot{x}_i = g_i(t,\mathbf{x},\mathbf{u}), \qquad i = 1,\ldots n, \\
&\qquad \mathbf{x}(0) = \overline{\mathbf{x}}, \\
&\qquad x_i(T) = x_{iT}, \quad i = 1,\ldots p \\
&\qquad x_i(T) \geq x_{iT}, \quad i = p+1,\ldots q \\
&\qquad x_i(T) \text{ free} \quad\;\; i = q+1,\ldots n. \\
&\qquad \mathbf{u} \in U \subseteq R^n
\end{aligned}
\tag{2.80}
$$

Theorem

For **u*** *and the resulting state vector* **x*** *to maximize* (2.80) *it is necessary that there exists a constant* λ_0 *and continuous functions* λ(t) (λ_i(t) i=1,...,n) *such that* $\forall t \in [0,T]$

$$\lambda_0 = 0 \text{ or } 1, \{\lambda_0, \lambda(t)\} \neq \{0,\mathbf{0}\}, \tag{2.81}$$

$$\mathbf{u}^* = \arg\max_{\mathbf{u}} H\left(t, \mathbf{x}^*, \mathbf{u}, \lambda\right), \tag{2.82}$$

where

$$H(t,\mathbf{x},\mathbf{u},\lambda) = \lambda_0 f(t,\mathbf{x},\mathbf{u}) + \sum_{i=1}^{n} \lambda_i g_i(t,\mathbf{x},\mathbf{u}) \tag{2.83}$$

except at points of discontinuity of **u**

$$\dot{\lambda}_i = -H_{x_i} \tag{2.84}$$

and the transversality conditions

$$\begin{aligned} &\lambda_i(T) \text{ free}, && i = 1, \ldots p, \\ &\lambda_i(T) \geq 0, && i = p+1, \ldots q, \\ &\lambda_i(T) = 0, \text{if } x_i(T) > x_{iT}, i = p+1, \ldots q, && \\ &\lambda_i(T) = 0 && i = q+1, \ldots n. \end{aligned} \tag{2.85}$$

The strange shadow value on the objective function λ_0 may under some perverse circumstances be 0. I believe you can safely ignore this possibility for the coming courses in economics.

Note that by specifying the control region we may formulate *Kuhn-Tucker* first order conditions instead of (2.82).

At the points in time when the control jumps λ has a kink. It is, however, *always continuous.* Note also that H is always continuous. Kuhn-Tucker shadow values on the control constraints, μ_t, may be discontinuous.