3. Dynamic Optimization in Discrete Time

3.1. Non-Stochastic Dynamic Programming

Consider the dynamic problem

$$\begin{aligned} &\max_{c,k} \sum_{t=0}^{T} \beta^t U(c_t) \\ &s.t. \quad k_0 = \bar{k}_0, \\ &\qquad\ \ k_{t+1} = f(k_t) - c_t,\ t = 0,\ldots,T, \\ &\qquad\ \ k_{T+1} = 0 \end{aligned} \tag{3.1}$$

The direct way to solve this would be to form the Lagrangean

$$L = \sum_{t=0}^{T} \beta^t U(c_t) + \sum_{t=0}^{T} \lambda_t \left((f(k_t) - c_t) - k_{t+1} \right) \tag{3.2}$$

with first order conditions

$$\begin{aligned} &\beta^t U'(c_t) - \lambda_t = 0, \\ &\lambda_t f'(k_t) - \lambda_{t-1} = 0. \end{aligned} \tag{3.3}$$

An alternative way is to recognize the recursive structure of the problem. (3.1) can be written

$$\begin{aligned} &\max_{c_0,k_1|_{k_0}} \left(U(c_0) + \beta \max_{c_1,k_2|_{k_1}} \left(U(c_1) + \ldots + \beta \max_{c_T,k_{T+1}|_{k_T}} \left(U(c_T) \right) \right) \right) \\ &s.t. \quad k_0 = \bar{k}_0, \\ &\qquad\ \ k_{t+1} = f(k_t) - c_t,\ t = 0,\ldots,T, \\ &\qquad\ \ k_{T+1} = 0 \end{aligned} \tag{3.4}$$

We then solve the problem backwards starting from the last period. In period T-1 the remaining problem only depends on earlier actions through k_{T-1}. Substituting from the constraint we then want to solve

$$\max_{k_T} U(f(k_{T-1}) - k_T) + \beta U(f(k_T)) \tag{3.5}$$

We need to solve for the *function* $k_T = h(k_{T-1})$, i.e., for all possible values of k_{T-1}. Then we solve the problem for *T-2* given that we do what is optimal in *T-1*. So we solve

$$\max_{k_{T-1}} U(f(k_{T-2}) - k_{T-1}) + \beta\left(\max_{k_T} U(f(k_{T-1}) - k_T) + \beta U(f(k_T))\right) \tag{3.6}$$

Define the current value function for the last periods problem as

$$V_1(k_{T-1}) = \max_{k_T} U(f(k_{T-1}) - k_T) + \beta U(f(k_T)). \tag{3.7}$$

Substitute into (3.6) and continue in the same iterative way to get

$$V_2(k_{T-2}) \equiv \max_{k_{T-1}} U(f(k_{T-2}) - k_{T-1}) + \beta V_1(k_{T-1}) \tag{3.8}$$

$$V_s(k_{T-s}) \equiv \max_{k_{T-s+1}} U(f(k_{T-s}) - k_{T-s+1}) + \beta V_{s-1}(k_{T-s+1}) \tag{3.9}$$

where *s* denote the number of periods left to termination.

The equations in (3.8) and (3.9) are called *Bellman equations*. The first order conditions implicitly defines difference equations for *k*.

$$-U'(f(k_{T-s}) - k_{T-s+1}) + \beta V'_{s-1}(k_{T-s+1}) = 0. \tag{3.10}$$

To find the policy functions $k_{T-s+1} = h_s(k_{T-s})$ we need to find the value function. In a finite horizon problem this is done as above by starting from the last period.

Infinite Horizon

In an infinite horizon problem we cannot use the method of starting from the last period. Instead we can use to different approaches. 1. Guess on a value function and make sure it satisfies the Bellman equation. 2. Iterate on the Bellman equation until it converges.

Guessing

Guessing is often feasible when the problem is autonomous. Then the value function is independent of time so we can write.

$$\begin{aligned} &V(k_t) = \max_{u_t} U(k_t, u_t) + \beta V(k_{t+1}) \\ &s.t. k_{t+1} = g(k_t, u_t). \end{aligned} \tag{3.11}$$

We can rewrite

$$V(k_t) = \max_{u_t} U(k_t, u_t) + \beta V(g(k_t, u_t)) \tag{3.12}$$

with first order conditions

$$U_u(k_t, u_t) + \beta V'(k_{t+1}) g_u(k_t, u_t) = 0. \tag{3.13}$$

Suppose we find a solution to (3.12) (and to (3.13) when relevant) has to be a function $u(k_t)$. Plugging that into (3.12) we get rid of the max so we have

$$V(k_t) = U(k_t, u_t(k_t)) + \beta V(g(k_t, u_t(k_t))) \tag{3.14}$$

Note that we can use the envelope result that we can evaluate $V'(k)$ as the partial derivative holding u constant, i.e.,

$$\begin{aligned} V'(k_t) &= U_k(k_t, u_t) + \frac{du_t}{dk_t} \left(\overbrace{U_u(k_t, u_t) + \beta V'(k_{t+1}) g(k_t, u_t)}^{=0} \right) \\ &\quad + \beta V'(g(k,u)) g_k(k,u) \\ &= U_k(k_t, u_t) + \beta V'(g(k,u)) g_k(k,u) \end{aligned} \quad . \tag{3.15}$$

If (3.14) is satisfied we have a solution to the value function. On the other hand, if, for example, u depends on more variables than k, (3.14) is not satisfied and our guess was incorrect.

Note that the whole RHS of (3.11) is a functional of the unknown function $V(.)$ for the given functions U and g. We can define this functional as

T(*V*). The Bellman equation then defines a *fixed point* for *T* in the space of *functions V*. The Bellman equation can thus be written

$$V(k) = T(V(k)) \equiv \max_u U(k,u) + \beta V(g(k,u)) \tag{3.16}$$

and a fixed point argument (contraction mapping) can be used to determine existence and uniqueness. Note that it is a fixed point *in the space of functions* we are looking for, not a fixed point for *k*. I.e., if we plug in some function of *k* in the RHS of (3.14) we must get out the same function on the LHS.

Typically the value function is of a similar for to the objective function. This is intuitive in the light of (3.14). For example if the utility function in (3.1) is logarithmic we guess that the value function is of the form $A \ln k + B$ for some constants *A*,*B*. For HARA utility functions (e.g., CRRA, CARA and quadratic) the value functions are generally of the same type as the utility function (Merton, 1971).

Iteration

An alternative way is try to find the limit of finite horizon Bellman equation as the horizon goes to infinity. Under for economical purposes quite general conditions this limit exists and is equal to the value function for the infinite horizon problem

$$\lim_{s \to \infty} V_s(k_{T-s}) \equiv V(k) \tag{3.17}$$

Using the notation in (3.16) we apply the operator T until the sequence converges.

$$V(k) = \lim_n T^n V(k) \tag{3.18}$$

if the limit exists. In this case we can be sure that this satisfies the Bellman equation which when we use the formulation in (3.16) and the definition in (3.18) becomes

$$\begin{aligned} &V(k) = T(V(k)) \\ &\lim_n T^n V(k) = T \lim_n T^n V(k) \\ &\lim_n T^n V(k) = \lim_n T^{n+1} V(k) \end{aligned} \tag{3.19}$$

With discounting it is typically unimportant what we plug in for $V_0(k)$ in (3.17). We can then start with any function and iterate until we get convergence. This can easily be done numerically, either by specifying a functional form, if we know that, or by just choosing a grid. In the latter case we just a set of values for the state variable $\{k_0, k_1, \ldots, k_n\}$. $V_0(k)$ is then a set of preliminary values (numbers) for each of the state variables in the grid.

An Iteration Example

In (3.1) let $U(c)=\ln(c)$ and $f(k)= k^{\alpha}$ with $0<\alpha<1$.We then have

$$\begin{aligned} &V_1(k_{T-1}) = \max_{k_T} \ln\left(k_{T-1}^{\alpha} - k_T\right) + \beta \ln k_T^{\alpha} \\ &FOC \qquad \frac{1}{\left(k_{T-1}^{\alpha} - k_T\right)} = \frac{\beta\alpha}{k_T} \Rightarrow k_T = \frac{\beta\alpha k_{T-1}^{\alpha}}{1+\alpha\beta} \end{aligned} \tag{3.20}$$

Substitute into the value function

$$\begin{aligned} V_1(k_{T-1}) &= \ln\left(\frac{1}{1+\alpha\beta} k_{T-1}^{\alpha}\right) + \beta \ln\left(\frac{k_{T-1}^{\alpha}}{1+\alpha\beta}\right)^{\alpha} \\ &= -(1+\alpha\beta)\ln(1+\alpha\beta) + \alpha(1+\alpha\beta)\ln k_{T-1} \end{aligned} \tag{3.21}$$

Then

$$
\begin{aligned}
& V_2(k_{T-2}) = \max_{k_{T-1}} \ln\left(k_{T-2}^{\alpha} - k_{T-1}\right) + \beta V_1(k_{T-1}) \\
& FOC \qquad \frac{1}{k_{T-2}^{\alpha} - k_{T-1}} = \beta V_1'(k_{T-1}) = \beta\alpha(1+\alpha\beta)\frac{1}{k_{T-1}} \qquad (3.22) \\
& \Rightarrow k_{T-1} = \frac{\beta\alpha(1+\alpha\beta)}{1+\beta\alpha(1+\alpha\beta)} k_{T-2}^{\alpha}
\end{aligned}
$$

So

$$
\begin{aligned}
V_2(k_{T-2}) &= \ln\left(\frac{k_{T-2}^{\alpha}}{1+\beta\alpha(1+\alpha\beta)}\right) + \beta V_1\left(\frac{\beta\alpha(1+\alpha\beta)}{1+\beta\alpha(1+\alpha\beta)} k_{T-2}^{\alpha}\right) \\
&= \alpha\left(1+\alpha\beta+\alpha^2\beta^2\right)\ln k_{T-2} \\
&\quad +\left(\beta+\alpha\beta+\alpha^2\beta^2\right)\ln\alpha\beta \qquad (3.23) \\
&\quad +(\alpha\beta+\alpha^2\beta^2-\beta-\beta^2)\ln(1+\alpha\beta) \\
&\quad -(1+\alpha\beta+\alpha^2\beta^2)\ln(1+\alpha\beta+\alpha^2\beta^2).
\end{aligned}
$$

It is easy to see that the coefficient on k is a power series that converge to $\alpha/(1-\alpha\beta)$ when the horizon goes to infinity (provided $\alpha\beta < 1$). Also the constants converge if also $0 < \beta < 1$ and the resulting function is

$$
\begin{aligned}
& \lim_{s\to\infty} V_s(k_{T-s}) \\
& \equiv V(k) = \frac{\alpha}{1-\alpha\beta}\ln k + \frac{\ln(1-\alpha\beta)}{1-\beta} + \frac{\alpha\beta\ln\alpha\beta}{(1-\beta)(1-\alpha\beta)} \qquad (3.24) \\
& = A\ln k + B.
\end{aligned}
$$

From this we can derive the optimal policy function by using the first order condition for the Bellman equation

$$V(k_t) = \max_{k_{t+1}} \ln\left(k_t^{\alpha} - k_{t+1}\right) + \beta V(k_{t+1})$$

$$FOC \qquad \frac{1}{k_t^{\alpha} - k_{t+1}} = \beta V'(k_{t+1}) = \frac{\beta A}{k_{t+1}} \tag{3.25}$$

$$\Rightarrow k_{t+1} = \frac{\beta A}{1+\beta A} k_t^{\alpha}$$

Note that the policy function is a stable difference equation under the assumptions about α,β.

Verification of Guess

If we had guessed the form $A \ln k + B$ the Bellman equation had become

$$V(k_t) = \max_{k_{t+1}} U\left(k_t^{\alpha} - k_{t+1}\right) + \beta V(k_{t+1}) \tag{3.26}$$

giving first order conditions

$$\begin{aligned} &U'\left(k_t^{\alpha} - k_{t+1}\right) = \beta V'(k_{t+1}) \\ &\frac{1}{k_t^{\alpha} - k_{t+1}} = \beta A \frac{1}{k_{t+1}} \Rightarrow k_{t+1} = \frac{A\beta}{1+A\beta} k_t^{\alpha} \end{aligned} \tag{3.27}$$

Plugging this into the Bellman equation yields

$$\begin{aligned} A \ln k_t + B &= \ln\left(k_t^{\alpha} - \frac{A\beta}{1+A\beta} k_t^{\alpha} \right) + \beta\left(A \ln \frac{A\beta}{1+A\beta} k_t^{\alpha} + B \right) \\ &= \alpha \ln k_t + \ln \frac{1}{1+A\beta} + \alpha\beta A \ln k_t + \beta A \ln \frac{A\beta}{1+A\beta} + \beta B \\ &= (\alpha + \alpha\beta A) \ln k_t + \ln \frac{1}{1+A\beta} + \beta A \ln \frac{A\beta}{1+A\beta} + \beta B \end{aligned} \tag{3.28}$$

which is satisfied if we set A and B to the values in (3.24).

State Variables

We often solve the dynamics programming problem by guessing a form of the value function. The first thing to determine is then which variables should enter, i.e., which variables are the state variables. The state variables must satisfy both following conditions

1. To enter the value function at time they must be realized at *t*.

   Note, however, that it sometimes may be convenient to use $E_t\,(z_{t+s})$ as a state variable. The expectation as of *t* is certainly realized at *t* even if the stochastic variable is not realized.

2. The set of variables chosen as state variables must together give sufficient information so that the value of the program from *t* and onwards when the optimal control is chosen can be calculated.

What do we need if the per period utility function in (3.1) was $U(c_t, c_{t-1})$?

Note, we should try to find the smallest such set. Look for example on the following problem.

$$\begin{aligned}
&\max_{c,k} \sum_{t=0}^{T} \beta^t U(c_t) \\
&s.t. \quad k_0 = \bar{k}_0, \\
&\qquad\quad l_0 = \bar{k}_0 \\
&\qquad\quad k_{t+1} + l_{t+1} = f(k_t, l_t) - c_t,\ t = 0, \ldots, T, \\
&\qquad\quad k_{T+1} = 0
\end{aligned} \tag{3.29}$$

In general we need both *k*, and *l* in the value function but if *f* is linear we may only need a linear combination. If $f(k_t, l_t) = a(k_t + l_t)$ we could define a new state variable $w = k+l$ and use $V(w)$ as our value function. The reason is that to compute the value of the program we only need to know the sum of *k* and *l*, their share are superfluous information.

3.2. Stochastic Dynamic Programming

As long as the recursive structure of the problem is intact adding a stochastic element to the transition equation does not change the Bellman equation. Consider the problem

$$\begin{aligned}
&\max_{\{u_t\}_0^\infty} E\sum_{t=0}^{\infty}\beta^t r(k_t,u_t)\\
&s.t. \quad k_0=\bar{k}_0,\\
&\qquad k_{t+1}=g(k_t,u_t,\varepsilon_{t+1}),\ \forall t\geq 0.\\
&\qquad \varepsilon_t=\begin{cases}\bar{\varepsilon}, & \text{with probability } p\\ \underline{\varepsilon}, & \text{with probability } (1\text{-}\ p)\end{cases}
\end{aligned} \tag{3.30}$$

where E is the expectations operator. Note that we have to specify the set of information that u_t can be conditioned on. Clearly it will in general be optimal to condition for example consumption on observed realizations of ε_t. If the agent may condition on information available at t we get the Bellman equation with first order conditions

$$\begin{aligned}
&V(k_t)\equiv\max_{u_t}\left\{r(k_t,u_t)+\beta\left[pV(g(k_t,u_t,\bar{\varepsilon}))+(1-p)V(g(k_t,u_t,\underline{\varepsilon}))\right]\right\}\\
&FOC\\
&r_u(k_t,u_t)+\\
&\beta\left[pV'(g(k_t,u_t,\bar{\varepsilon}))g_u(k_t,u_t,\bar{\varepsilon})+(1-p)V'(g(k_t,u_t,\underline{\varepsilon}))g_u(k_t,u_t,\underline{\varepsilon})\right]\\
&=0
\end{aligned} \tag{3.31}$$

or for a general distribution F of ε

$$\begin{aligned}
&V(k_t)\equiv\max_{u_t}\left\{r(k_t,u_t)+\beta EV(g(k_t,u_t,\bar{\varepsilon}))\right\}\\
&FOC \qquad r_u(k_t,u_t)+\beta E\left(V'(g(k_t,u_t,\varepsilon))g_u(k_t,u_t,\varepsilon)\right)=0
\end{aligned} \tag{3.32}$$

where E denotes the expectations operator. Note that $V(k_t)$ in (3.31) and (3.32) is a current value function.

A Stochastic Consumption Example

Consider the following program

$$
\begin{aligned}
&\max_{c,\omega} \sum_{t=0}^{\infty} \beta^t \ln c_t \\
&s.t. \quad A_{t+1} = (A_t - c_t)\big((1+r)\omega + (1+z_t)(1-\omega)\big).
\end{aligned} \tag{3.33}
$$

The consumer decides how much to consume each period. The share ω of here assets is placed in a riskless asset yielding *r* in return and (1-ω) in a risky asset with return z_t, that is i.i.d.

The problem is autonomous so we write the current value Bellman equation with time independent value function *V*

$$
V(A_t) = \max_{c_t,\omega}\Big[\ln c_t + \beta E_t V\big((A_t - c_t)\big((1+r)\omega + (1+z_t)(1-\omega)\big)\big)\Big] \tag{3.34}
$$

Necessary first order conditions yield

$$
\begin{aligned}
&c_t; \quad \frac{1}{c_t} - \beta E_t V'(A_{t+1})\big((1+r)\omega + (1+z_t)(1-\omega)\big) = 0, \\
&\omega_t; \quad E_t V'(A_{t+1})(A_t - c_t)(r - z_t) = 0.
\end{aligned} \tag{3.35}
$$

Now we use Merton's result and guess that the value function is

$$
V(A_t) = a \ln A_t + B \tag{3.36}
$$

for some constants *a* and *B*. Substituting into (3.35) we get

$$
\begin{aligned}
&\frac{1}{c_t} = \beta E_t a \frac{1}{A_{t+1}}\big((1+r)\omega + (1+z_t)(1-\omega)\big) \\
&= \beta a \frac{1}{(A_t - c_t)} \Rightarrow c_t = \frac{A_t}{1+a\beta}.
\end{aligned} \tag{3.37}
$$

and

$$
\begin{aligned}
&E_t V'(A_{t+1})(A_t - c_t)(r - z_t) \\
&= E_t \frac{\cancel{(A_t - c_t)}(r - z_t)}{\cancel{(A_t - c_t)}\big((1+r)\omega + (1+z_t)(1-\omega)\big)} \\
&= E_t \frac{(r - z_t)}{\big((1+r)\omega + (1+z_t)(1-\omega)\big)} = 0.
\end{aligned} \tag{3.38}
$$

Note that (3.38) implies that ω is constant since z_t is i.i.d.

Now we have to solve for the constant a. This is done by substituting the solutions to the first order conditions and the guess into the Bellman equations.

$$
\begin{aligned}
&a \ln A_t + B \\
&= \ln A_t - \ln(1+a\beta) + \beta E_t (a \ln A_{t+1} + B) \\
&= \ln A_t - \ln(1+a\beta) + \beta a \ln(A_t - c_t) + \beta B \\
&\ +\beta a E_t \ln\big((1+r)\omega + (1+z_t)(1-\omega)\big) \\
&= \ln A_t - \ln(1+a\beta) + \beta a(\ln A_t + \ln a\beta - \ln(1+a\beta)) \\
&\ +\beta B + \beta a E_t \ln\big((1+r)\omega + (1+z_t)(1-\omega)\big) \\
&= (1+a\beta)\ln A_t + k \\
&\Rightarrow a = \frac{1}{1-\beta}, \qquad c_t = (1-\beta) A_t
\end{aligned} \tag{3.39}
$$

where k is some constant. This also verifies that the guess worked.

3.3. Stochastic Dynamic Optimization in Continuous Time

Dynamic Programming in Continuous Time

Assume that $J(x)$ is the optimal current value function for the following problem.

$$\max_u \int_0^\infty e^{-rt} f(x(t),u(t))dt$$
$$s.t. \quad \dot{x} = g(x(t),u(t)) \tag{3.40}$$

If the problem is time consistent we can apply Bellman's principle of optimality and split the problem as in the discrete time case. For an (infinitely) small time interval dt we then get

$$J(x_t) = \max_{u_t}\left[f(x_t,u_t)dt + e^{-rdt} J(x_{t+dt})\right]$$
$$s.t. \quad x_{t+dt} = x_t + g(x_t,u_t)dt \tag{3.41}$$

Now approximate $e^{-rdt} \approx 1 - rdt$, and use the first order Taylor approximation $J(x_{t+dt}) \approx J(x_t) + J'(x_t)(dx/dt)dt$ to get

$$J(x_t) = \max_{u_t}\left[f(x_t,u_t)dt + (1-rdt)\left(J(x_t) + J'(x_t)g(x_t,u_t)dt\right)\right]$$
$$\Rightarrow rJ(x_t)\cancel{dt} = \max_{u_t}\left[f(x_t,u_t)\cancel{dt} + J'(x_t)g(x_t,u_t)(dt - \cancel{rdt^2})\right] \tag{3.42}$$
$$\Rightarrow rJ(x_t) = \max_{u_t}\left[f(x_t,u_t) + J'(x_t)g(x_t,u_t)\right]$$

where I have used the fact that dt^2 is very (infinitely) small relative to dt to approximate it by zero.

A Well-Known Example Again

Assume the consumer solves

$$\max_c \int_0^\infty e^{-\theta t} \ln(c)dt$$
$$s.t. \quad \dot{k} = rk - c. \tag{3.43}$$

The Bellman equation is

$$
\begin{aligned}
&\theta J(k) = \max_c \left[\ln c + J'(k)(rk - c)\right] \\
&F.O.C. \qquad \frac{1}{c} - J'(k) = 0.
\end{aligned} \tag{3.44}
$$

Now guess that the form of the value function is $A \ln k + B$. Use this and the solution to the first order condition to get

$$
\begin{aligned}
&\theta(A \ln k + B) = \ln k - \ln A + \frac{A}{k}\left(rk - \frac{k}{A}\right) \\
&= \ln k - \ln A + Ar - 1 \\
&\Rightarrow A = \theta^{-1}, B = \frac{r - \theta + \theta \ln \theta}{\theta^2}
\end{aligned} \tag{3.45}
$$

We can then easily find the consumption function

$$
c_t = \theta k_t \Rightarrow \frac{\dot{c}_t}{c_t} = \frac{\dot{k}_t}{k_t} = r - \theta \tag{3.46}
$$

We can check the results for the value function for the case r=θ. Then consumption is equal to rk so $J(k)=\ln rk/r$. This is consistent with (3.46).

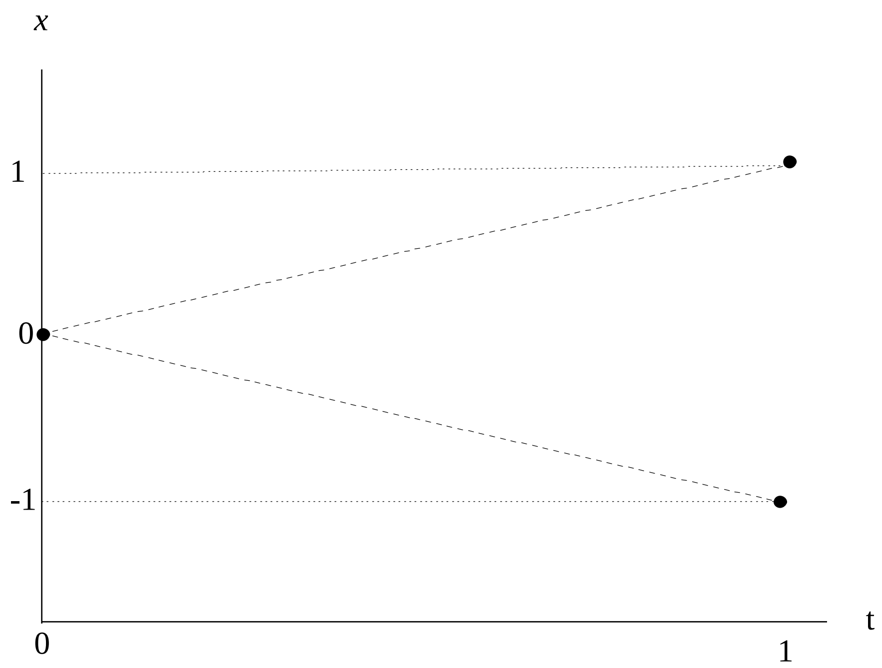
Brownian Motion and Itô's Lemma

An Heuristic Description of a Wiener Process

Consider a discrete binomial process $z(t)$ that during a unitary time interval takes a positive jump of height 1 with probability 0.5 and negative jump of height 1 with probability 0.5. I.e,

$$
\begin{aligned}
&z(0) = 0, \\
&z(1) = \begin{cases} 1 & \text{with probability 0.5} \\ -1 & \text{with probability 0.5.} \end{cases}
\end{aligned} \tag{3.47}
$$

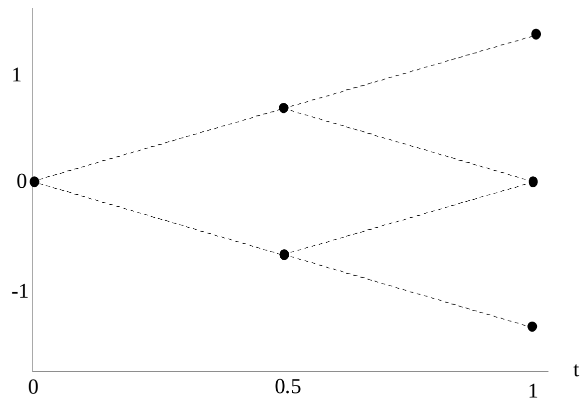
The variance per unit of time is then 1.

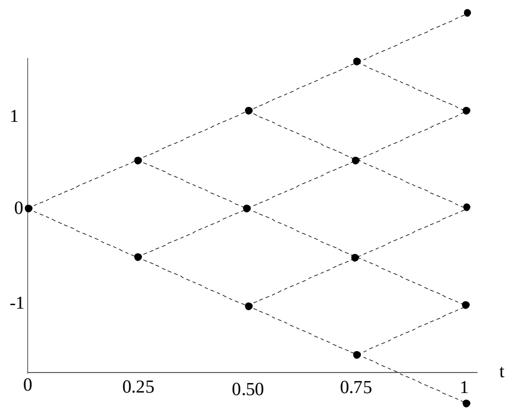


Now divide the time interval [0,1] into n sub-intervals, each of length $\Delta t_n = 1/n$ and let the process jump at each sub-interval. The direction of the jump is independent of earlier jumps. We then adjust the amplitude s_n in each jump so that *the variance per unit of time is held constant.* I.e., we want

$$\frac{0.5s_n^2 + (1-0.5)s_n^2}{\Delta t_n} = \frac{s_n^2}{\Delta t_n} = 1 \tag{3.48}$$

For a partition in 2 and 4 sub-intervals the process can be described as in the graphs.





If the time interval is divided into n partitions the variable z(1) is distributed as a Binomial (0.5,n). From the central limit theorem we know that this is approximately a standard normal distribution if n is large. Furthermore $z(0.5)$ is approximately a normal with variance 0.5. The same is true for the variable ($z(1)$-$z(0.5)$) which also is independent of $z(0.5)$.

More generally, increments of z over time intervals that are large relative to the partition are approximately normal with variance equal to the length of the time interval. Increments over disjoint time intervals are furthermore independent.

Now look at how the amplitude s_n relative to the length of the time interval Δt_n change when n increases. From (3.48) we have that

$$\begin{aligned} &\frac{s_n^2}{\Delta t_n} = \frac{s_n^2}{1/n} = 1, \Rightarrow s_n = \sqrt{1/n} \\ &\Rightarrow \frac{s_n}{\Delta t_n} = \frac{\sqrt{1/n}}{1/n} = \sqrt{n}. \end{aligned} \tag{3.49}$$

So we find that the relative amplitude is strictly monotonically increasing in n. For technical reasons there are complications involved in letting n go to infinity. But suppose we could, we would then guess that the resulting stochastic variable has the following properties.

$$z(0) = 0, \tag{3.50}$$

$$(z(t_1) - z(t_0)) \overset{d}{=} N(0, t_1 - t_0), \qquad \forall t_1 > t_0 \geq 0, \tag{3.51}$$

$$\begin{aligned} &(z(t_3) - z(t_2)), (z(t_1) - z(t_0)) \text{ independent} \\ &\forall, t_3 > t_2 > t_1 > t_0 \geq 0 \end{aligned}, \tag{3.52}$$

$$z \text{ is continuous in time,} \tag{3.53}$$

$$z \text{ is nowhere differentiable.} \tag{3.54}$$

(3.53) is understandable since the jumps s_n would go towards zero when n increases and (3.54) since the *relative* jump size continues to grow as n increases.

The properties (3.50)-(3.54) are the properties of a *Wiener Process.*

If we think of new information as coming in a continuous flow we should model it as a *Wiener* process. It has become popular to model stochastic variables that are continuous and directly related to information flows as following a modified Wiener process called a *Brownian motion* that typically is written as

$$dx = \mu(t,z,\omega)dt + \sigma(t,z,\omega)dz \tag{3.55}$$

A typical example is a stock market index. In (3.55) we have allowed for a non-stochastic but variable time trend. σ multiplies the Wiener process to allow for different flow rates at different times.

Itô's Lemma

Suppose we are trying to find a value function $J(x)$ for a problem where the state variable x follows a Brownian motion. To use the dynamic programming approach we need to calculate how J evolves over time. For this we need *Itô's Lemma*

$$\begin{aligned} &dx = \mu(t,z,\omega)dt + \sigma(t,z,\omega)dz \\ &\Rightarrow dJ(x) = J'(x)\mu dt + J'(x)\sigma dz + \frac{1}{2}J''(x)\sigma^2 dt \end{aligned} \tag{3.56}$$

We can think of this in the following way. Since the Wiener process is so extremely non-smooth we need to do a second order Taylor approximation, the ususal first order that we use for continuous functions is not good enough.

A way to remember *Itô's Lemma* is to make a second order Taylor approximation and use the rule that

$$
\begin{aligned}
dz^2 &= dt, \\
dt^2 &= 0, \\
dzdt &= 0.
\end{aligned} \tag{3.57}
$$

So

$$
\begin{aligned}
dx &= \mu(t,z,\omega)dt + \sigma(t,z,\omega)dz \\
\Rightarrow dJ(x) &= J'(x)dx + \frac{1}{2}J''(x)dx^2 \\
&= J'(x)\mu dt + J'(x)\sigma dz + \frac{1}{2}J''(x)\mu^2 dt^2 \\
&\quad + J''(x)\mu dt \sigma dz + \frac{1}{2}J''(x)\sigma^2 dz^2 \\
&= J'(x)\mu dt + J'(x)\sigma dz + \frac{1}{2}J''(x)\sigma^2 dt
\end{aligned} \tag{3.58}
$$

A Stochastic Consumption Example

Consider the continuous time variant of (3.33)

$$
\begin{aligned}
&J(W_s) = \max_{c,\omega} \int_s^{\infty} e^{-r(t-s)} U(c_t)dt \\
&s.t. \quad dW_t = W_t\left(\omega r_0 dt + (1-\omega)(r_1 dt + \sigma dz_t)\right) - c_t dt \\
&\qquad W_s \text{ given.}
\end{aligned} \tag{3.59}
$$

The consumer decides how much to consume each point in time. The share ω of her financial assets is placed in a riskless asset yielding $r_0 dt$ in return and (1-ω) in a risky asset with return ($r_1 dt+\sigma dz$). Using a variant of (3.42) the Bellman equation can be written

$$
\begin{aligned}
&rJ(W) = \max_{c,\omega}\left[U(c) + \frac{E\left(dJ(W)\right)}{dt}\right] \\
&s.t. \quad dW_t = W_t\left(\omega r_0 dt + (1-\omega)(r_1 dt + \sigma dz_t)\right) - c_t dt
\end{aligned} \tag{3.60}
$$

Using Itô's lemma and the computational rules in (3.57) we have that

$$
\begin{aligned}
dJ &= dWJ' + \frac{1}{2} dW^2 J'' \\
&= \left(W_t\left(\omega r_0 dt + (1-\omega)(r_1 dt + \sigma dz_t)\right) - c_t dt\right) J' \\
&\quad + \frac{1}{2}\left(W_t\left(\omega r_0 dt + (1-\omega)(r_1 dt + \sigma dz_t)\right) - c_t dt\right)^2 J'' \\
&= \left(W_t\left(\omega r_0 dt + (1-\omega)(r_1 dt + \sigma dz_t)\right) - c_t dt\right) J' \\
&\quad + \frac{1}{2}\left(W_t (1-\omega)\sigma\right)^2 dtJ''.
\end{aligned}
\tag{3.61}
$$

and

$$
\frac{EdJ}{dt} = \left(W_t\left(\omega r_0 + (1-\omega) r_1\right) - c_t\right) J' + \frac{1}{2}\left(W_t (1-\omega)\sigma\right)^2 J''. \tag{3.62}
$$

The first order conditions for (3.60) then becomes

$$
\begin{aligned}
&U'(c_t) - J' = 0 \\
&(r_0 - r_1)\cancel{W_t} J' - (1-\omega) W_t^{\cancel{2}} \sigma^2 J'' = 0.
\end{aligned}
\tag{3.63}
$$

Guessing that $J(W) = A\ln(W) + B$ we get

$$
\begin{aligned}
&c = \frac{W}{A} \\
&(r_0 - r_1)\cancel{\frac{A}{W}} + (1-\omega)\cancel{W}\sigma^2 \frac{A}{\cancel{W^2}} = 0 \\
&\Rightarrow (1-\omega) = \frac{r_1 - r_0}{\sigma^2}
\end{aligned}
\tag{3.64}
$$

So as in the discrete case consumption is a constant share of wealth and the share of assets held in the risky asset increase in the premium r_1 -r_0 and decrease in risk

By substituting the solutions into the Bellman equation we get

$$
\begin{aligned}
&r(A\ln W+B)\\
&=\ln W-\ln A+\left(W\left(\omega r_0+(1-\omega)r_1\right)-\overbrace{\frac{W}{A}}^{c}\right)\overbrace{\frac{A}{W}}^{J'}\\
&+\frac{1}{2}\left(W(1-\omega)\sigma\right)^2\overbrace{\frac{A}{-W^2}}^{J''} \qquad (3.65)\\
&=\ln W\underbrace{-\ln A+\left(A\left(\omega r_0+(1-\omega)r_1\right)-1\right)-\frac{1}{2}\left((1-\omega)\sigma\right)^2 A}_{B}
\end{aligned}
$$

This verifies that the guess was correct and that A=1/r as in the discrete time case.