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OPTIMAL FISCAL AND MONETARY POLICY IN AN ECONOMY WITHOUT CAPITAL*

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This paper is concerned with the structure and time-consistency of optimal fiscal and monetary policy in an economy without capital. In a dynamic context, optimal taxation means distributing tax distortions over time in a welfare-maximizing way. For a barter economy, our main finding is that with debt commitments of sufficiently rich maturity structure, an optimal policy, if one exists, is time-consistent. In a monetary economy, the idea of optimal taxation must be broadened to include an 'inflation tax', and we find that time-consistency does not carry over. An optimal 'inflation tax' requires commitment by 'rules' in a sense that has no counterpart in the dynamic theory of ordinary excise taxes. The reason time-consistency fails in a monetary economy is that nominal assets should, from a welfare-maximizing point of view, always be taxed away via an immediate inflation in a kind of 'capital levy'. This emerges as a new possibility when money is introduced into an economy without capital.

1. Introduction

This paper is an application of the theory of optimal taxation to the study of aggregative fiscal and monetary policy. Our analysis is squarely in the neoclassical, welfare-economic tradition stemming from Ramsey's (1927) contribution, so it will be useful to begin by reviewing the leading applications of this theory to aggregative questions of public finance, and by situating our approach and results within this tradition.

Ramsey studied a static, one ('representative') consumer economy with many goods. A government requires fixed amounts of each of these goods, which are purchased at market prices, financed through the levy of flat-rate excise taxes on the consumption goods. It is assumed that for any given pattern of excise taxes, prices and quantities are established competitively. In this setting, Ramsey sought to characterize the excise tax pattern(s) that

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would maximize the utility of the consumer (or minimize the 'excess burden' or 'welfare cost' of taxation). He thus abstracted from distributional questions and from issues of possible conflict between the objectives of 'government' and those governed, abstractions that will be maintained in this paper, as they were in those cited below.

Pigou (1947) and later Kydland and Prescott (1977), Barro (1979), Turnovsky and Brock (1980), and others noted that Ramsey's formulation could be applied to the study of fiscal policy over time if the many goods being taxed were interpreted as dated deliveries of a single, aggregate consumption good. In this reinterpretation, the excise tax on 'good t' is interpreted as the general level of taxes in period t. Since tax receipts in a given period will not, in general, be optimally set equal to government consumption in that period, the theory of optimal taxation becomes, in this reinterpretation, a theory of the optimal use of public debt as well. Roughly concurrently, Bailey (1956), Friedman (1969), Phelps (1973), Calvo (1978) and others developed the observation that if one could interpret the holding of cash balances as consumption, at each date, of a second 'good' then the Ramsey formulation could be applied to the study of monetary as well as fiscal policy, with the 'inflation tax' induced by monetary expansions playing the formal role of an ordinary excise tax.

In all of these applications of the Ramsey theory, tax rates on various goods are thought of as being simultaneously chosen. In Ramsey's original static setting this assumption seems a natural one, but in a dynamic application it is more realistic to think of tax rates as being set sequentially through time by a succession of governments, each with essentially no ability to bind the tax decisions of its successor governments. Kydland and Prescott (1977) showed, through a series of graphic examples, how fundamental a difference this reinterpretation makes. If government at each date is free to rethink the optimal tax problem from the current date on, it will not, in general, find it best to continue with the policy initially found to be optimal. In the terminology of Strotz (1955–1956), tax policies optimal in the Ramsey sense are, in general, *time-inconsistent*. Since the normative advice to a society to follow a specific 'optimal' policy is operational only if that policy might conceivably be carried out over time under the political institutions within which that society operates, the Kydland–Prescott paper calls into serious question the applicability of all dynamic adaptations of the Ramsey framework.

One 'reason' for the time-inconsistency of optimal policies is the classical issue of the 'capital levy'. In the Ramsey framework, with lump-sum (and hence non-distorting) taxes assumed unavailable, it is best to focus excise taxes on goods that are inelastically supplied or demanded, to tax 'pure rents'. In a dynamic setting, goods produced in the past, capital, always have this quality and the returns to such goods are thus 'optimally' taxed away.

Yet it will clearly not induce an optimal pattern of capital accumulation if such confiscatory taxes are announced for the future. Such a discrepancy between the best future tax policies to announce today and the best policy actually to execute when the future arrives is precisely what is meant by time-inconsistency.

In the present paper, we consider only economies without capital of any form, so that the difficult issues raised by capital levies are simply set aside. Private and government consumption goods are assumed to be produced under constant returns to scale using labor as the only input, and government consumption is taken to follow an exogenously given stochastic process. Moreover, the analysis is conducted in a neoclassical framework, thus precluding any countercyclical role for fiscal or monetary policy.

In section 2 we consider a barter economy. We assume that in each period the current government has full control over current tax rates, the issue of new debt, and the refinancing (at market prices) of old debt. However, it takes as fully binding the debt commitments made by its predecessors. We ask whether debt commitments (fully honored) are sufficient to induce successor governments to continue — *as if* they were bound to do so — tax policies that are optimal initially or sufficient, in short, to enforce the time-consistency of optimal tax policies. Our main finding is that with debt commitments of a sufficiently rich maturity structure an optimal policy, if one exists, can be made time consistent. That is, given an optimal tax policy, there exists a unique debt policy that makes it time-consistent. Section 3 consists of a series of examples, in which optimal tax-debt policies are characterized for a variety of specific assumptions about government consumption.

In section 4, money is introduced, its use motivated by a Clower (1967)-type transactions demand, modified to permit velocity to be responsive to variations in interest rates. Within this framework, familiar results on the optimal 'inflation tax' are readily replicated by exploiting the analogies between this monetary economy and the barter economy studied in section 2. With respect to the time-consistency of optimal policies, however, these analogies turn out, perhaps not surprisingly, to be more misleading than helpful. An optimal 'inflation tax' requires commitment by 'rules' in a sense that does not seem to have a counterpart in the dynamic theory of ordinary excise taxes.

Section 5 contains an informal discussion of the likely consequences of relaxing some of the simplifying assumptions of our necessarily abstract treatment of these issues, and of some directions on which further progress might be made. Section 6 is a compact summary of the main findings.

2. A Barter economy

Though the issues raised in the introduction have mainly to do with monetary economies, it is convenient to begin with the study of fiscal policies in a simple barter economy. In this section, we describe one such economy, and characterize the equilibrium behavior of prices and quantities in the economy for a *given* fiscal policy. With this as a background, alternative ways of formulating the problem faced by the government will then be discussed.

There is one produced good, and government consumption of this good is taken to follow a given stochastic process, the realizations $g \equiv (g_0, g_1, g_2, \ldots)$ of which have the joint distribution F.[1] Let F^t denote the marginal distribution of the history $g^t \equiv (g_0, g_1, \ldots, g_t)$ of these shocks from 0 through t, for $t = 0, 1, 2, \ldots$. Assume that F has a density f, and let f^t denote the density for F^t. Finally, define $g_s^t \equiv (g_s, g_{s+1}, \ldots, g_t)$, for $0 \leqq s \leqq t$, and let $F_s^t(\cdot \,|g^{s-1})$, with density $f_s^t(\cdot \,|g^{s-1})$, denote the conditional distribution of g_s^t given g^{s-1}. (Evidently, these distributions will need to be restricted to assure that feasible patterns of government consumption exist. We postpone the question of how this might best be done.)

There is no other source of uncertainty in the economy, so that the basic commodity space will be the space of infinite sequences $(c, x) = \{(c_t, x_t)\}_{t=0}^{\infty}$, where c_t, private consumption of the produced good in period t, and x_t, private consumption of 'leisure' in period t, are both (contingent-claim) functions of g^t, the history of government shocks between 0 and t. Prices, tax rates, and government obligations, all to be introduced below, will lie in this same space. The endowment of labor in each period is unity, the produced good is non-storable, and the technology is such that one unit of labor yields one unit of output, so that feasible allocations are those satisfying

$$c_t + x_t + g_t \leqq 1, \qquad t = 0, 1, 2, \ldots, \quad \text{all } g^t. \tag{2.1}$$

The preferences of the single, 'representative' consumer are then given by the von Neumann–Morgenstern utility function

$$E\left\{\sum_{t=0}^{\infty} \beta^t U(c_t, x_t)\right\} = \sum_{t=0}^{\infty} \beta^t \int U(c_t(g^t), x_t(g^t))\,\mathrm{d}F^t(g^t). \tag{2.2}$$

The discount factor β is between 0 and 1, and the current period utility function, $U: R_+^2 \to R$, is strictly increasing in both arguments and strictly concave, with goods and leisure both normal (non-inferior).

[1] Many, perhaps most, of the main points made below could as well have been developed in a context of perfect certainty [as in Turnovsky and Brock (1980)] so there is something to be said for the strategy of simply reading 'z' wherever we write '$\int z\,\mathrm{d}F^t(g^t)$' or '$\int z\,\mathrm{d}g^t$'. The reader for whom this simplification is helpful is invited to do this. When we turn, in section 3, to characterizing optimal fiscal policies under erratic government expenditure paths, however, the stochastic examples seem easier to interpret than the deterministic ones.

Since there is no capital in this system, it is clear that *efficient* allocations (c, x) are fully characterized by (2.1) and the condition

$$U_c(c_t, x_t) = U_x(c_t, x_t), \quad t = 0, 1, 2, \ldots, \quad \text{all } g^t, \tag{2.3}$$

to the effect that the marginal rate of substitution between goods and leisure is equal to the marginal rate of transformation, unity. If lump-sum taxes were available, the optimal policy would be to set the tax in period t equal to g_t, so that (2.3) would always hold. We will assume, to the contrary, that the *only* tax available to the government is a flat-rate tax τ_t levied against labor income $1 - x_t$. Under a continuously balanced government budget, then, the equality $g_t = \tau_t(1 - x_t)$ would hold each period, under all realizations of g^t.

To admit other possibilities, we will introduce government debt (possibly negative), in the form of sequences ${}_tb = \{{}_tb_s\}_{s=t}^{\infty}$, $t = 0, 1, 2, \ldots$, where ${}_tb_s(g^{t-1}, g_t^s)$ is the claim held by the consumer at the beginning of period t, given that the event g^{t-1} occurred, to consumption goods in period $s \geq t$, contingent on the event g_t^s. The idea of a government issuing contingent claims may seem an odd one, but it is easy to introduce into the formalism we are using and it permits us, as will be seen below, to consider fiscal policies of practical interest that could not be analyzed if government debt were assumed at the outset to represent a certain claim on future goods.

The market structure throughout will be as follows. In each period $t = 0, 1, 2, \ldots$, from the point of view of both the government and the representative consumer, current and past government expenditures, g^t, are known: future government expenditures g_{t+1}^{∞} are given by 'nature', with known conditional distribution $F_{t+1}^{\infty}(\cdot|g^t)$; and the consumer's contingent claims to current and future goods, ${}_tb$, are given by history. Given g^t, there are markets for the current consumption good $c_t(g^t)$ and current labor $x_t(g^t)$, and a complete set of securities markets for future contingent claims, ${}_{t+1}b_s(g^t, g_{t+1}^s)$, $s = t+1, t+2, \ldots$, all g_{t+1}^s. Given these market arrangements, we examine in turn the optimal behavior of consumers for given prices and taxes, the determination of competitive equilibrium, given taxes and government spending, and finally the optimal behavior of the fiscal authority. All questions of characterizing optimal fiscal policies under various assumptions on the shocks g will be deferred to the next section.

2.1. Consumer behavior

First, consider the behavior of the representative consumer. Assume that he takes as given the sequence $\tau = \{\tau_t\}_{t=0}^{\infty}$ of contingent tax rates, and the price sequence $p = \{p_t\}_{t=0}^{\infty}$, where $p_t(g^t)$ is interpreted as follows. The consumer (correctly) expects that in each period $t = 0, 1, 2, \ldots$, given g^t, the market price of a claim to a unit of current goods or labor will be $p_t(g^t)$ and the market price of a contingent claim to a unit of goods in period s, contingent on the event g^s_{t+1}, will be $p_s(g^t, g^s_{t+1})$, $s=t+1,\ t+2,\dots$, all g^s_{t+1}.

The consumer's behavior is described in two stages. In period $t=0$, given τ, p, F, and g_0, the consumer solves his optimization problem by *planning* a sequence of (contingent) consumptions of goods and leisure, (c,x). However, in the market in each period $t=0,1,2,\dots$, he *trades* only current goods and labor (c_t, x_t), and assets, $\{_{t+1}b_s\}^{\infty}_{s=t+1}$. Consequently he must be careful to carry out these trades in such a way that he will in fact be able to afford to purchase his planned allocation in every period t, for every realization of g^t.

The consumer's planning problem, then, is to maximize (2.2), with τ, p, F, and g_0 given, subject to the budget constraint

$$p_0[c_0-(1-\tau_0)(1-x_0)-{}_0b_0]+\sum_{t=1}^{\infty}\int p_t[c_t-(1-\tau_t)(1-x_t)-{}_0b_t]\,\mathrm{d}g^t_1\leqq 0. \tag{2.4}$$

The first-order conditions for this concave program are (2.4), with equality, and (if the solution is interior) the marginal conditions

$$\frac{U_x(c_t,x_t)}{U_c(c_t,x_t)}=1-\tau_t,\qquad t=0,1,2,\dots,\quad \text{all } g^t_1,\quad \text{and} \tag{2.5}$$

$$\beta^t\frac{U_c(c_t,x_t)}{U_c(c_0,x_0)}f^t_1(g^t_1|g_0)=\frac{p_t}{p_0},\qquad t=0,1,2,\dots,\quad \text{all } g^t_1. \tag{2.6}$$

Let (c,x) be the solution of (2.4)–(2.6), given (τ,p). (Since U is strictly concave, the solution will be unique.)

The transactions required to attain this allocation are carried out as follows. When the market meets in period t, with g^t known, the consumer *purchases* his current allocation $(c_t(g^t), x_t(g^t))$, and any bond holdings $_{t+1}b$ satisfying

$$\begin{aligned}&p_{t+1}\,{}_{t+1}b_{t+1}+\sum_{s=t+2}^{\infty}\int p_s\,{}_{t+1}b_s\,\mathrm{d}g^s_{t+2}\\&=p_{t+1}[c_{t+1}-(1-\tau_{t+1})(1-x_{t+1})]+\sum_{s=t+2}^{\infty}\int p_s[c_s-(1-\tau_s)(1-x_s)]\,\mathrm{d}g^s_{t+2},\end{aligned} \tag{2.7}$$

all g_{t+1}, g^t given.

This ensures that his budget constraint in the following period will be satisfied, for any realization of g_{t+1}. The consumer is indifferent among all bond holdings ${}_{t+1}b$ satisfying (2.7). To see that the required bond holdings are always in the consumer's budget set, suppose that (2.7) holds for some particular g_t, g^{t-1} given. Then choose any ${}_{t+1}b$ satisfying (2.7) for (g^t, g_{t+1}), all g_{t+1}, g^t given. Integrating the second set of equations with respect to g_{t+1} and subtracting the first from it one obtains

$$p_t[c_t-(1-\tau_t)(1-x_t)-{}_tb_t]+\sum_{s=t+1}^{\infty}\int p_s[{}_{t+1}b_s-{}_tb_s]\mathrm{d}g_{t+1}^s=0,$$

so that the chosen bond holdings ${}_{t+1}b$ are in the consumer's budget set at g^t. Thus, by induction, if (2.7) holds at g^t, the required debt holdings of the consumer are affordable at all later dates. Since (2.7) holds for $t=-1$ [cf. (2.4)], the argument is complete.

2.2. Competitive equilibrium

With consumer behavior thus described, given τ and F an *equilibrium* resource allocation plan (c,x) — if one exists — is uniquely determined from (2.1) and (2.5), with supporting prices (interest factors), p, given in (2.6). Substituting from (2.5) and (2.6) into (2.4) and simplifying, one sees that the following condition must hold in a competitive equilibrium:

$$\begin{aligned}&(c_0-{}_0b_0)U_c(c_0,x_0)-(1-x_0)U_x(c_0,x_0)\\&+\sum_{t=1}^{\infty}\beta^t\int[(c_t-{}_0b_t)U_c(c_t,x_t)-(1-x_t)U_x(c_t,x_t)]\,\mathrm{d}F_1^t(g_1^t|g_0)=0.\end{aligned}\tag{2.8}$$

From the government's point of view in period 0, given current government consumption, g_0, given the conditional distribution of future government consumption, F_1^{∞}, and given the existing (contingent) government obligations, ${}_0b$, any allocation (c,x) that can be implemented by some tax policy τ must thus satisfy (2.1) and (2.8). Conversely, any allocation that satisfies (2.1) and (2.8) can be implemented by setting tax rates according to (2.5). Equilibrium prices, given those tax rates, are described by (2.6), and the required debt restructurings $\{{}_tb\}_{t=1}^{\infty}$ are any sequence satisfying (2.7) for $t=0,1,2,\ldots$. Eqs. (2.1) and (2.8) then provide a complete description of the set of competitive equilibrium allocations attainable through feasible government policies.

Note that by Walras' law, if eq. (2.4) holds then the government budget constraint is also satisfied. Substituting from (2.1), one finds that (2.4) is simply a statement to the effect that the present value of outstanding government obligations must equal the present value of the excesses of tax revenues over government expenditures on goods. Writing this familiar condition in the form (2.8) emphasizes the facts that the choice of a tax policy in effect dictates the private sector equilibrium resource allocation and, in particular, dictates the interest rates to be used in carrying out this present value calculation. It is for the latter reason that one cannot take the initial *value* of government debt as historically given to the current government. One needs to know the entire schedule of (contingent) coupon payments due.

2.3. Optimal fiscal policy with commitment

With the behavior of the private sector, given a fiscal policy, spelled out in (2.5)–(2.8), we turn to the problem faced by government in choosing a fiscal policy. Here and throughout the paper we take the *objective* of government to be to maximize consumer welfare as given in expression (2.2). As is well known, this hypothesis is consistent with a variety of equilibria, depending on what is assumed about the government's ability to bind itself (or its successors) at time 0 to state-contingent decisions that will actually be carried out at times $t>0$. We will initially consider the problem faced by a government with the ability to bind itself at time 0 to a tax policy for the entire future. Later on, we will ask whether such a policy might actually be carried out under a more realistic view of government institutional arrangements.

Define, then, an *optimal* (tax-induced) allocation $(c,x)=\{(c_t,x_t)\}$ as one that maximizes (2.2) subject to (2.1) and (2.8). Letting λ_0 be the multiplier associated with the constraint (2.8), and $\mu_{0t}(g^t)\geqq 0$ be the multiplier associated with (2.1) for g^t, the first-order conditions for this problem are (2.1), (2.8) and

$$(1+\lambda_0)U_c+\lambda_0[(c_t-{}_0b_t)U_{cc}+(x_t-1)U_{cx}]-\mu_{0t}=0, \tag{2.9a}$$

$$t=0,1,2,\ldots,\quad \text{all } g^t,$$

$$(1+\lambda_0)U_x+\lambda_0[(c_t-{}_0b_t)U_{cx}+(x_t-1)U_{xx}]-\mu_{0t}=0, \tag{2.9b}$$

where the derivatives of U are evaluated at (c_t,x_t). Since the second-order conditions for this maximization problem involve third derivatives of U, solutions to (2.1), (2.8)–(2.9) may represent local maxima, minima, or saddle points. Or, (2.1), (2.8)–(2.9) may have no solution. Clearly, if g and/or ${}_0b$ are 'too large', there will be no feasible policy (no policy satisfying the government's budget constraint), and hence no optimal policy. However, assuming — as we will — that an optimal policy exists and that the solution is interior, it will satisfy (2.1), (2.8)–(2.9). Our analysis applies to these situations only. Appendix A treats the issues of existence and uniqueness of an optimal policy for an example with quadratic utility.

To construct a solution to (2.1), (2.8)–(2.9), one would solve (2.1) and (2.9) for c_t and x_t as functions of $g^t, {}_0b_t$, and λ_0, and then substitute these functions into (2.8) to obtain an equation in the unknown λ_0. Having so obtained the optimal allocation (c, x), the tax policy τ that will implement it is given in (2.5) and the resulting equilibrium prices p in (2.6).

In each period $t=0,1,2,\ldots$, debt issues or retirements will be required to make up the difference between current tax revenue, $\tau_t(1-x_t)$, and the sum of current government consumption and current debt payments due, $g_t + {}_tb_t$. Thus, the government must in each period buy or sell bonds at market prices, and do this in such a way that the end-of-period debt, ${}_{t+1}b$, satisfies (2.7). However, it is clear that once the government is committed to a particular tax policy for all time, relative prices of traded commodities and securities at each date are determined, so that within the constraint imposed by (2.7), only the *total* value of the debt at these prices matters. That is, *given* current and future tax rates, the maturity structure of the debt is of no consequence, provided that (2.7) holds.

2.4. *Time consistency of the optimal fiscal policy*

The optimal tax policy given implicitly in (2.1), (2.8)–(2.9) is of interest as a benchmark, but the decision problem it solves has no clear counterpart in actual democratic societies. In practice, a government in office at time t is free to re-assess the tax policy selected earlier, continuing it or not as it sees fit. To study fiscal policies that might actually be carried out under institutional arrangements bearing some resemblance to those that now exist, we need to face up to the problem of time-inconsistency. There are many ways to do this; we choose the following.

Imagine the government at $t=0$ as choosing the current tax rate, τ_0, announcing a future tax policy $\{\tau_t\}_{t=1}^{\infty}$, and restructuring the outstanding debt, leaving the government at $t=1$ with the maturity structure ${}_1b$. Take this debt-restructuring to be carried out at prices consistent with the announcements of future tax policies being perfectly credible. Imagine the government at $t=1$ to be fully bound to honor the debt ${}_1b$, but to be free to select any current tax rate τ_1' it wishes, to announce any future taxes $\{\tau_t'\}_{t=2}^{\infty}$ it wishes, and to restructure the debt as it wishes. The debt restructuring at $t=1$ is carried out at prices consistent with the *new* announcements $\{\tau_t'\}_{t=2}^{\infty}$ being perfectly credible. Suppose that the (contingent) tax rates announced at $t=0$ are always chosen at $t=1$, $\tau_1 \equiv \tau_1'$, all g^1, and that the (contingent) tax rates for subsequent periods announced at $t=0$ are announced again at $t=1$, $\tau_t \equiv \tau_t'$, $t=2,3,\ldots$, all g^t. Suppose, moreover, that this is true for all later periods as well. Then we will call the optimal policy *time-consistent*.

As shown in Appendix B, if the optimal policy is time-consistent in this sense, it is also time-consistent in the following (weaker) sense: The policy (current tax rate and debt restructuring as functions of current government consumption and inherited debt) of each dated government, maximizes that government's objective function (the total discounted expected utility of the consumer from the current period on), taking as given the (maximizing) policies to be adopted by its successors. This holds for every possible value of the state variables (current government consumption and inherited debt), for every dated government. Viewing the dated governments as players in a game, a time-consistent optimal policy corresponds to a set of subgame perfect Nash equilibrium strategies (one for each player).

Somewhat surprisingly, we will show that *the optimal policy is time-consistent.*[2] More exactly, we show that if an allocation (c,x) together with a multiplier λ_0 satisfy (2.1), (2.8)–(2.9), then it is always possible to choose a restructured debt $\{{}_1b_t\}_{t=1}^{\infty}$, at market prices given by (2.6), such that the continuation $\{(c_t,x_t)\}_{t=1}^{\infty}$ of this same allocation satisfies (2.1), (2.8)–(2.9), given ${}_1b$, for all realizations g^1. By induction, then, the same is true in all later periods.

If such a ${}_1b$ can be chosen, there must be functions $\lambda_1(g^1)$ and $\mu_{1t}(g^t)$, such that

$$\sum_{t=1}^{\infty} \beta^t \int [(c_t - {}_1b_t)U_c - (1-x_t)U_x] \mathrm{d}F^t(g^t|g^1) = 0, \quad \text{all } g^1, \tag{2.8'}$$

$$(1+\lambda_1)U_c + \lambda_1[(c_t - {}_1b_t)U_{cc} + (x_t - 1)U_{cx}] - \mu_{1t} = 0. \tag{2.9a'}$$

$$t = 1,2,3,\ldots, \text{ all } g^t,$$

$$(1+\lambda_1)U_x + \lambda_1[(c_t - {}_1b_t)U_{cx} + (x_t - 1)U_{xx}] - \mu_{1t} = 0, \tag{2.9b'}$$

hold at $\{(c_t,x_t)\}_{t=1}^{\infty}$. Since by assumption leisure is a normal good, $U_{cc} - U_{cx} < 0$. Therefore, adding (2.9a) minus (2.9b) minus (2.9a') plus (2.9b'), and solving for ${}_1b_t$ for each fixed $t \geqq 1$ and g^t gives

$$\lambda_1\,{}_1b_t = \lambda_0\,{}_0b_t + (\lambda_1 - \lambda_0)a_t, \qquad t = 1,2,3,\ldots, \quad \text{all } g^t, \quad \text{where} \tag{2.10}$$

$$a_t(g^t) \equiv [(U_c - U_x) + (U_{cc} - U_{cx})c_t + (U_{xx} - U_{cx})(1-x_t)]/(U_{cc} - U_{cx}), \qquad t = 1,2,3,\ldots, \quad \text{all } g^t. \tag{2.11}$$

If $\lambda_0 = 0$, then from (2.9) and (2.9') we see that $\lambda_1 = 0$. If $\lambda_0 \neq 0$, then $\lambda_1 \neq 0$, and substituting for ${}_1b$ from (2.10) into (2.7) yields an equation in λ_1 that has a unique solution for each g^1; the resulting values for ${}_1b$ satisfy (2.8').

[2]This conclusion differs from that reached by Turnovsky and Brock (1980), in a context very similar to this one. The key difference is that our formulation involves debt issues at all maturities, while theirs restricts attention to one-period debt only. It is easy to see that the time-consistency proof below fails if the restriction ${}_tb_s = 0$ for $s > t$ is added.

The following example illustrates why the maturity structure of the debt is important. Let the utility function be quadratic:

$$U(c,x)=c+x-\tfrac{1}{2}(c^2+x^2), \quad \text{so that}$$

$$U_c=1-c, \quad U_x=1-x, \quad U_{cc}=U_{xx}=-1, \quad U_{cx}=0.$$

Then combining (2.9a) and (2.9b) to eliminate μ_{0t}, at an optimum:

$$(1+\lambda_0)(x_t-c_t)-\lambda_0[c_t-{}_0b_t+(1-x_t)]=0, \qquad t=0,1,2.$$

Let there be three periods, $t=0,1,2$, and let $\beta=1$. Suppose that there is no government consumption, $g_0=g_1=g_2=0$, and that there is a constant amount of debt due in each period, ${}_0b_0={}_0b_1={}_0b_2=\frac{1}{6}$. Therefore, substituting from (1), necessary conditions for an optimum are

$$(1+\lambda_0)(1-2c_t)-\lambda_0[2c_t-\tfrac{1}{6}]=0, \qquad t=0,1,2.$$

Thus, $c_0=c_1=c_2$, so that (2.8) requires

$$(c_t-\tfrac{1}{6})(1-c_t)-c_t^2=0, \qquad t=0,1,2.$$

The relevant solution (see appendix A) is

$$\tau_t=\tfrac{1}{2}, \qquad c_t=\tfrac{1}{3}, \qquad x_t=\tfrac{2}{3}, \qquad p_t=1, \qquad t=0,1,2.$$

Taxing at the optimal rate at $t=0$ generates exactly enough revenue to redeem the currently maturing debt, and the optimal debt policy is to leave the existing (flat) maturity structure in place: ${}_1b_1={}_1b_2=\frac{1}{6}$. Clearly the optimal plan is time-consistent under this restructuring: when the government at $t=1$ optimizes it will choose $\tau_1=\frac{1}{2}$, the revenue collected will exactly cover debt currently due, and the debt due at $t=2$ will be left in place. The government at $t=2$ will set $\tau_2=\frac{1}{2}$, and redeem the remaining debt.

Now suppose instead that the government at $t=0$ were to restructure the debt, at the prices $p_1=p_2=1$, so that it was all long term, ${}_1b'_1=0$ and ${}_1b'_2=\frac{1}{3}$. Then in period $t=1$, necessary conditions for an optimum would be

$$(1+\lambda_1)(1-2c_1)-\lambda_1[2c_1]=0, \quad (1+\lambda_1)(1-2c_2)-\lambda_1[2c_2-\tfrac{1}{3}]=0.$$

Clearly these will not be satisfied with $c_1=c_2$. Instead the optimum is (approximately)

$$c'_1\approx 0.38, \quad \tau'_1\approx 0.53, \quad c'_2\approx 0.32, \quad \tau'_2\approx 0.39, \quad p'_2/p'_1\approx 0.91.$$

Note that by raising the current tax rate and lowering the future tax rate, the government at $t=1$ induces an increase in current goods consumption and a fall in future goods consumption. This is accompanied by a fall in the price of future goods relative to current goods, i.e., a rise in the interest rate. Thus, the value of the outstanding debt, measured in goods at $t=1$, falls. It is this 'devaluing' of the debt that provides an incentive for the (benevolent) government at $t=1$ to deviate from the optimal (at $t=0$) tax policy. (Note that if consumers foresee this, they will not exchange short-term for long-term debt on a one-for-one basis at $t=0$.)

2.5. Extension to many consumer goods

It is not difficult to extend this formulation, the calculation of the optimal open-loop allocation, and the above time-consistency conclusion, to the case of many non-storable consumption goods. Since this extension turns out to be useful in the analysis (section 4) of a monetary economy, we will develop it briefly here. Let there be n produced goods, so that period t's consumption is the vector $c_t=(c_{1t},\dots,c_{nt})$, and the description (2.1) of the technology is replaced by

$$\sum_{i=1}^{n} c_{it}+x_t+g_t \leqq 1. \tag{2.12}$$

Preferences are given by (2.2), but with c_t reinterpreted as an n-vector so that $U: R_+^{n+1}\to R$. The consumer's budget constraint (2.4) is replaced by

$$\begin{aligned} &p_0\left[1-x_0-\sum_{i=1}^{n}(1+\theta_{i0})(c_{i0}-b_{i0})\right] \\ &+\sum_{t=1}^{\infty}\int p_t\left[1-x_t-\sum_{i=1}^{n}(1+\theta_{it})(c_{it}-b_{it})\right]\mathrm{d}g_1^t=0, \end{aligned} \tag{2.13}$$

where $\theta_{it}(g^t)$ is a state-contingent excise tax levied on good i in state g^t.

Notice that in (2.13), in contrast to (2.4), goods purchases, not labor sales, are taxed. The one good case studied above corresponds here to the case $n=1$, with $1+\theta_{it}\equiv(1-\tau_t)^{-1}$. This is a notational modification only. Notice also that there are n types of contingent bonds in (2.13), one for each good, and that the coupon payments b_{it} on these bonds are not subject to tax.[3] Notice finally that if 'leisure' could be taxed symmetrically with the other n goods in the system, then taxing the $n+1$ 'goods' $c_{1t},\dots,c_{nt}$ and x_t at a

[3]This argument for making interest payments on government debt non-taxable was anticipated, in an early recognition of the importance of time-consistency, by Hamilton (1795).

common rate would be the equivalent of a direct tax on the endowment, or of a lump-sum tax. Eq. (2.13) is written in a way that rules out this possibility. These last two remarks point up substantive features of this formulation that are crucial to the conclusions that follow.

The first-order conditions for the problem: maximize (2.2) subject to (2.13), are (2.13),

$$\beta^t \frac{U_x(c_t, x_t)}{U_x(c_0, x_0)} f^t(g^t) = \frac{p_t}{p_0}, \qquad t=0,1,2,\dots, \text{ all } g^t, \text{ and} \tag{2.14}$$

$$\frac{U_i(c_t, x_t)}{U_x(c_t, x_t)} = 1 + \theta_{it}, \qquad i=1,2,\dots,n, \quad t=0,1,2,\dots, \text{ all } g^t, \tag{2.15}$$

where $U_i(c_t, x_t) = (\partial/\partial c_{it}) U(c_t, x_t)$. Letting $U' \equiv (U_1, U_2, \dots, U_n, U_x)^T$, any allocation (c, x) satisfying (2.12) and

$$\sum_{t=0}^{\infty} \beta^t \int \begin{bmatrix} c_t - {}_0b_t \\ x_t - 1 \end{bmatrix}^T \cdot U' \mathrm{d}F^t(g^t | g_0) = 0, \tag{2.16}$$

can be implemented using taxes only on goods $i = 1, \dots, n$. Prices are then given in (2.14), tax rates in (2.15).

An optimal open-loop tax policy, then, corresponds to an allocation (c, x) that maximizes (2.2) subject to (2.12) and (2.16). The first-order conditions for this problem, written with the arguments of U and its derivatives suppressed, are (2.12), (2.16) and

$$(1+\lambda_0)U' + \lambda_0 U'' \begin{bmatrix} c_t - {}_0b_t \\ x_t - 1 \end{bmatrix} - \mu_{0t}\mathbf{1} = \mathbf{0}, \qquad t=0,1,2,\dots, \text{ all } g^t, \tag{2.17}$$

where λ_0 is the multiplier associated with (2.16), $\mu_{0t}(g^t) \geqq 0$ is the multiplier associated with (2.12) for state g^t, and U'' is the matrix

$$U'' \equiv \frac{\partial^2 U}{\partial (c_t, x_t)^2} = \begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1n} & U_{1x} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ U_{n1} & U_{n2} & \cdots & U_{nn} & U_{nx} \\ U_{x1} & U_{x2} & \cdots & U_{xn} & U_{xx} \end{bmatrix}.$$

The $n+2$ equations in (2.17) and (2.12) correspond to (2.9) and (2.1) for the one-good case. Note that within each period, in each state, the optimal allocation satisfies the Ramsey tax rule, modified only for the existence of outstanding debt, ${}_0b_t \neq 0$. If ${}_0b_t(g^t)=0$, the optimal tax rates $\theta_i(g^t)$, $i=1,2,\dots,n$, are the usual Ramsey taxes.[4]

Constructing an optimal tax policy involves, then, the following steps. First, solve (2.17) and (2.12) for the allocations (c_t, x_t) as functions of g^t, ${}_0b_t$, and λ_0. Insert these functions into (2.16) to obtain λ_0, and hence the optimal allocation. Finally, use (2.15) to obtain the excise tax structure that implements this allocation.

The definition of time-consistency used in the one-good case serves as well for the many-goods case under examination here, and the proof that the optimal open-loop policy is time-consistent involves no new elements. Premultiplying (2.17) by the $n\times(n+1)$ matrix $[I_n \vdots -\mathbf{1}]$ to eliminate μ_{0t}, and subtracting the analogous system of equations for period 1, we find that

$$(\lambda_0-\lambda_1)[I_n \vdots -\mathbf{1}]\left[U'+U''\begin{bmatrix} c_t \\ x_t-1\end{bmatrix}\right]-[I_n \vdots -\mathbf{1}]U''\begin{bmatrix}\lambda_0\,{}_0b_t-\lambda_1\,{}_1b_t \\ 0\end{bmatrix}=\mathbf{0}. \tag{2.18}$$

Since by assumption leisure is a normal good, the $n\times(n+1)$ matrix $[I_n \vdots -\mathbf{1}]U''$ has rank n, so that ${}_1b_t$ is uniquely given by

$$\lambda_1\,{}_1b_t=\lambda_0\,{}_0b_t+(\lambda_1-\lambda_0)a_t, \qquad t=1,2,\dots,\ \text{all } g^t, \tag{2.19}$$

where a_t is the (unique) solution of

$$[I_n \vdots -\mathbf{1}]U''\begin{bmatrix} a_t \\ 0\end{bmatrix}=[I_n \vdots -\mathbf{1}]\left[U'+U''\begin{bmatrix} c_t \\ x_t-1\end{bmatrix}\right], \qquad t=1,2,\dots,\text{all } g^t. \tag{2.20}$$

[4]The connection with standard Ramsey taxes is most clearly seen as follows. Define (c^*,x^*) by

$$U_1(c^*,x^*)=U_2(c^*,x^*)=\cdots=U_n(c^*,x^*)=U_x(c^*,x^*), \quad \textstyle\sum_j c_j^*+x^*-1=0.$$

and let δ be the common value of $U_i(c^*,x^*)$. Then for g_t and ${}_0b_t$ small, or whenever U is a quadratic form, we can write

$$U'\approx\delta\mathbf{1}+U''^*\begin{bmatrix} c_t-c^* \\ x_t-x^*\end{bmatrix},$$

where U''^* is the matrix U'' evaluated at (c^*,x^*). Note that since U is strictly concave, U''^* is an $(n+1)\times(n+1)$ matrix of full rank. Substituting into (2.17) and approximating U'' by U''^*, we find that

$$(1+\lambda_0)U''^*\begin{bmatrix} c_t-c^* \\ x_t-x^*\end{bmatrix}+\lambda_0U''^*\begin{bmatrix} c_t-{}_0b_t \\ x_t-1\end{bmatrix}+(\delta(1+\lambda_0)-\mu_{0t})\mathbf{1}=\mathbf{0}.$$

The solution $(c_t,x_t)^T\in R_+^{n+1}$ is unique, given μ_{0t}. The required value for μ_{0t} yields a satisfying (2.12).

2.6. Summary

It is worth re-emphasizing the *central* importance in this analysis of optimal fiscal policy over time of the nature of a government's ability to bind its successors. One sees from (2.1), (2.5) and (2.6) [or from (2.12), (2.14) and (2.15)] that *if* the government could commit itself at $t=0$ to a complete set of current and future contingent tax rates, this commitment would fully determine the equilibrium resource allocation and the associated equilibrium prices. If such a commitment were possible, the maturity and risk structure of the debt would be immaterial. This case of complete commitment lies at one extreme of the range of possibilities.

At the other extreme, one might imagine a government with *no* ability to commit its successors, so that any debt it issued would be honored by its successors if they found it in their interest to do so, and repudiated otherwise. In this case, it is evident from (2.7) or (2.16) that debt commitments reduce the set of feasible allocations, so that at time 0, a government with the ability simply to repudiate debt will always choose to do so. In this situation, of course, no debt could ever be sold to the public in the first place, so that in fact all government consumption would have to be financed out of contemporaneous taxes. In general, this allocation will be inferior to the optimal policy with debt available (in the sense of yielding lower expected utility).

Our analysis has been focused on a situation intermediate between these two, in which there are no binding commitments on future taxes but in which debt commitments are fully binding. Our interest in this case does not arise from features that are intrinsic to the theory, since the theory sheds no light on why certain commitments can be made binding and others not, but because this combination of binding debts and transient tax policies seems to come closest to the institutional arrangements we observe in stable, democratically governed countries. It would be interesting to know why this is so, but pursuit of this issue would take us too far afield.

Our main finding, for this intermediate situation, is that being unable to make commitments about future tax rates is not a constraint. In the absence of any ability to bind choices about tax rates directly, each government restructures the debt in a way that *induces* its successors to continue with the optimal tax policy. For this to be possible, a rich enough mix of debt instruments must be available, where 'rich enough' means, roughly, one security for each dated, state-contingent good being traded ('leisure' expected).

3. Characteristics of optimal fiscal policies

In the preceding section we obtained the necessary conditions for optimal fiscal policies, and showed that optimal policies are time-consistent. This analysis was carried out with the path of government expenditures and the initial pattern of inherited government debt permitted to take essentially any form. In this section we present several examples, in each restricting government expenditures and initial debt to a specific form, so that we can characterize more sharply the optimal resource allocation and associated tax and debt policies. The idea in the simpler examples is to build up confidence that what we are calling 'optimal policies' accord with common sense, and in the more complicated ones to learn something about how fiscal policy ought ideally to be conducted.

The following preliminary calculations will be useful in the examples. First, substitute from (2.1), (2.5) and (2.6) into (2.8) to get

$$\sum_{t=0}^{\infty} \beta^t \int U_c[\tau_t(1-x_t)-g_t-{}_0b_t]\mathrm{d}F^t(g^t|g_0)=0. \tag{3.1}$$

Then multiplying (2.9a) by $(c_t-{}_0b_t)$ and (2.9b) by (x_t-1) and summing, we find that

$$\begin{aligned}&(1+\lambda_0)[(c_t-{}_0b_t)U_c+(x_t-1)U_x]\\&+\lambda_0[(c_t-{}_0b_t)^2U_{cc}+2(c_t-{}_0b_t)(x_t-1)U_{cx}+(x_t-1)^2U_{xx}]\\&-(c_t+x_t-1-{}_0b_t)\mu_{0t}=0.\end{aligned} \tag{3.2}$$

Note that since U is strictly concave, the quadratic term in (3.2) is negative. Finally, integrating (3.2) with respect to $\mathrm{d}F^t(g^t)$, multiplying the tth equation by β^t, summing over t, and using (2.1) and (2.8), we find that

$$\lambda_0 Q+\sum_{t=0}^{\infty} \beta^t \int (g_t+{}_0b_t)\mu_{0t}\mathrm{d}F^t(g^t|g_0)=0, \tag{3.3}$$

where Q is the sum of negative terms. Since $Q<0$, and $\mu_{0t}>0$, $t=0,1,2,\ldots$, all g^t, it follows from (3.3) that if $(g_t+{}_0b_t)>0$, $t=0,1,2,\ldots$, all g^t, then $\lambda_0>0$.

In all of the examples that follow, we assume that g_0, F_1^{∞}, and ${}_0b$ are such that an optimal policy exists.

Example 1. Let $g\equiv 0$ and ${}_0b\equiv 0$. Since $Q<0$, it follows from (3.3) that $\lambda_0=0$. Hence (2.9) implies that the optimal allocation is constant over time, $(c_t,x_t)=(\bar{c},\bar{x})$, $t=0,1,2,\ldots$, where $(\bar{c},\bar{x})$ satisfies (2.1) and the efficiency condition $U_c(\bar{c},\bar{x})=U_x(\bar{c},\bar{x})$. From (2.5) it then follows that the optimal tax rates are identically zero, $\tau\equiv 0$.

Since the optimal policy is time-consistent, the analog of (2.9) must hold when the government re-solves its optimization problem in later periods.

Letting λ_t denote the multiplier associated with the analog of (2.8) in period t, this implies that $\lambda_t = \lambda_0 = 0$, $t = 1, 2, 3, \ldots$. Hence from (2.10), debt issues are indeterminate except that — from the government budget constraint — the *net* value of debt issues must be zero in each period.

Example 2. Let $g_t + {}_0b_t = 0$, $t = 0, 1, 2, \ldots$, all g^t. As in the previous example, it follows from (3.3) that $\lambda_0 = 0$. Hence, using (2.9), we find that the optimal allocation (c_t, x_t) is given by (2.1) and

$$U_c(c_t, x_t) = U_x(c_t, x_t), \qquad t = 0, 1, 2, \ldots, \quad \text{all } g^t.$$

The optimal tax and debt policies are exactly as in Example 1.

In Example 1 there is no government activity. In Example 2, the private sector initially holds a pattern of lump-sum obligations to government that precisely offset government consumption demand. In neither case is there any need to resort to distorting taxes, so that the multiplier λ_0 associated with the government budget constraint in each case is zero.

Example 3. Let $g_t = G$ and ${}_0b_t = B$, be constants for $t = 0, 1, 2, \ldots$, with $G + B > 0$. Then from (2.9), the optimal allocation is constant over time: $(c_t, x_t) = (\bar{c}, \bar{x})$, $t = 0, 1, 2, \ldots$, and from (2.5), the tax rate required to implement the optimal allocation is also constant over time: $\tau_t = \bar{\tau}$, $t = 0, 1, 2, \ldots$. Consequently, (3.1) implies that the government budget is balanced in each period, or that tax revenue in each period is just sufficient to cover current government consumption and redeem the currently maturing debt:

$$\bar{\tau}(1 - \bar{x}) - G - B = 0.$$

Since $G + B > 0$, it follows from (3.3) that $\lambda_0 > 0$. Since the analog of (2.9) must hold in all later periods, it follows that $\lambda_t = \lambda_0 > 0$, $t = 0, 1, 2, \ldots$. From (2.10) it then follows that no new debt is ever issued, and in each period only the currently maturing debt is redeemed, ${}_sb_t = B$, all s, t.

The function of government debt issues is to smooth distortions over time. If expenditures and debt obligations are smooth, as in this example, they are optimally financed from contemporaneous taxes. Nothing is gained either by issuing new debt or retiring existing debt.

Our remaining examples exploit the following simplification of (2.10). If the system begins with no debt outstanding, new issues of debt under the optimal policy have a particular form. Recall that if $\lambda_0 \neq 0$, then $\lambda_t \neq 0$, $t = 1, 2, \ldots$, all g^t. Assume that $\lambda_0 \neq 0$. If ${}_0b \equiv 0_s$, $s = 1, 2, 3, \ldots$, all g^s, then from

(2.10), in period 0 debt issues will be

$$ {}_1b_s=(1-\lambda_0/\lambda_1)a_s, \qquad s=1,2,\ldots, \quad \text{all } g^s, $$

where a_s is as defined in (2.11). In period 1 debt issues will be

$$ {}_2b_s=\frac{\lambda_1}{\lambda_2}\,{}_1b_s+\left(1-\frac{\lambda_1}{\lambda_2}\right)a_s=\left(\frac{\lambda_1}{\lambda_2}\left(1-\frac{\lambda_0}{\lambda_1}\right)+\left(1-\frac{\lambda_1}{\lambda_2}\right)\right)a_s $$

$$ =\left(1-\frac{\lambda_0}{\lambda_2}\right)a_s, \qquad s=2,3,\ldots, \quad \text{all } g^s. $$

Continuing by induction, one finds that if an optimal policy is followed from the beginning, then at any date t, the outstanding debt obligations satisfy

$$ {}_tb_s=(1-\lambda_0/\lambda_t)a_s, \qquad s=t,t+1,t+2,\ldots, \quad t=1,2,\ldots. \tag{3.4} $$

Thus, at the beginning of any period t, in any state g^t, there is in effect only one security outstanding — a bond of infinite maturity. The current coupon payment on this bond is $a_t(g^t)$, and the coupon payment in any period $s>t$, contingent on the event g^s_{t+1}, is $a_s(g^t,g^s_{t+1})$. The quantity of this security outstanding is $(1-\lambda_0/\lambda_t(g^t))$.

Therefore, in period $t-1$, an array of such securities — indexed by g_t — must be traded. Since the government in period $t-1$ inherits $(1-\lambda_0/\lambda_{t-1}(g^{t-1}))$ outstanding bonds (of infinite maturity), its securities trades must be as follows.

It meets the current coupon payments $(1-\lambda_0/\lambda_{t-1})a_{t-1}$ on the (single type of) outstanding bonds, and then buys all of those bonds back from consumers. At the same time it issues a new *set* of (contingent) bonds, each of which is contingent on the single event g_t, government consumption in the next period. For each possible value for g_t, it issues the quantity $(1-\lambda_0/\lambda_t(g^{t-1},g_t))$ of an infinite-maturity bond with the following coupon payments: $a_t(g^{t-1},g_t)$ in a period t, contingent on the event g_t; $a_t(g^{t-1},g_t,g^s_{t+1})$ in any period $s>t$, contingent on the joint event $[g_t$ and $g^s_{t+1}]$; and zero in all periods if g_t does not occur.

[Note that this holds for the many-goods case as well. If ${}_0b\equiv 0$, then there is a single security at the beginning of any period t, which is a bond of infinite maturity. The only difference is that the coupon payment on this bond in any period $s\geqq t$ is the *vector* of consumption goods, $a_s(g^s)$, defined in (2.20). Thus, with many goods, the single security is a type of indexed bond, where the index weights for each period s are contingent on the event g^s. As in the one-good case, during each period t, the government issues an array of securities, each contingent on the single event g_{t+1}.]

Values for $(1-\lambda_0/\lambda_t)$ can then be found by using (2.7), substituting from (2.6), and using (3.4).

$$\left(1-\frac{\lambda_0}{\lambda_t}\right)\left[U_c a_t+\sum_{t+1}^{\infty}\beta^{s-t}\int U_c a_s f_{t+1}^s \,\mathrm{d}g_{t+1}^s\right]$$

$$=U_c[c_t-(1-\tau_t)(1-x_t)]$$

$$+\sum_{t+1}^{\infty}\beta^s \int U_c[c_s-(1-\tau_s)(1-x_s)]f_{t+1}^s \,\mathrm{d}g_{t+1}^s \tag{3.5}$$

$$t=0,1,2,\ldots,\quad \text{all } g^t,$$

Example 4. Let ${}_0b\equiv 0$, $g_T>0$, and $g_t=0$ for $t\neq T$. From (2.9), the optimal allocation $(c_t,x_t)=(\bar{c},\bar{x})$ is co nt for all $t\neq T$, and consequently, from (2.5) and (3.4), the tax rate and coupon payment are also constant over these periods, $\tau_t=\bar{\tau}$, and $a_t=\bar{a}$, $t\neq T$. Using (3.2) we can study revenues. For $t\neq T$, $c_t+x_t-1-{}_0b_t=0$, and the last term in (3.2) drops out. Since $\lambda_0>0$, the second (quadratic) term is negative, so that the first term must be positive. Since $(1+\lambda_0)>0$, this implies

$$0<\bar{c}+(\bar{x}-1)U_x/U_c=\bar{c}+(\bar{x}-1)(1-\bar{\tau})=\tau(1-\bar{x}),$$

so that tax revenue is positive for $t\neq T$. For period T, the last term in (3.2), $\mu_T g_T$, is positive. Therefore, the sign of the first term is indeterminate: labor may be either taxed or subsidized in period T.

Consequently, debt issues are as follows. In each period $t=0,1,\ldots,T-1$, the government runs a surplus, using it to buy bonds issued by the private sector. In period T, the expenditure g_T is met by selling all of these bonds, possibly levying a tax on current labor income, and issuing new consols which have a coupon payment of $\bar{a}$ in every period. From (3.5) we see that

$$(1-\lambda_0/\lambda_t)=[\bar{c}-(1-\bar{\tau})(1-\bar{x})]/\bar{a},\qquad t=T+1,T+2,\ldots.$$

Hence $\lambda_t=\bar{\lambda}$, i. a constant for all $t\geqq T+1$, and (3.4) implies that a constant number of consols is outstanding in al periods $t\geqq T+1$. That is, in each period $t=T+1,T+2,\ldots$, tax revenue is just sufficient to service the interest on the outstanding consols, and none are ever redeemed.

Example 4 corresponds to a perfectly foreseen war, and is the most pointed possible illustration of the role of optimal fiscal policy in using debt to redistribute tax distortions over time. Note the symmetry over time, previously noted by Barro (1979): consumption is the same in all periods in which government expenditure is zero, regardless of the proximity to the date T at which the positive government expenditure g_T occurs.

Example 5. Let ${}_0b\equiv 0$, let $g_t=0$ for all $t\neq T$, and let $g_T=G>0$ with probability α and $g_T=0$ with probability $1-\alpha$. As in Example 4, $(c_t,x_t)=(\bar{c},\bar{x})$ (although the optimum values of $\bar{c}$ and $\bar{x}$ will not, in general, be the same) all $t\neq T$. In addition, (2.9) implies that $(c_T,x_T)=(\bar{c},\bar{x})$ if $g_T=0$. The argument in Example 4 shows that tax revenue is positive in all these states. Consequently, debt issues are as follows.

In periods $t=0,1,\ldots,T-2$, current tax revenue and interest income of the government are used to buy (infinite-maturity) bonds issued by the consumer. These bonds have a (certain) coupon payment of $\bar{a}$ in each period $t\neq T$; in period T they have a (contingent) coupon payment of $\bar{a}$ if $g_T=0$, and of $\hat{a}\neq\bar{a}$ if $g_T=G$.

In period $T-1$, the government collects current tax revenue and interest income, and sells back to the consumer all of its bond holdings. In addition, it issues 'contingent consols'; these have a coupon payment of $\bar{a}$ every period, payable if and only if $g_T=0$. All of these revenues are used to buy *from* consumers 'contingent bonds' of infinite maturity, which have a coupon payment of $\hat{a}$ in period T and $\bar{a}$ in every period thereafter, payable if and only if $g_T=G$.

In period T, if $g_T=0$, the consols held by the consumer have value, and the bonds held by the government do not. Tax revenue $\bar{\tau}(1-\bar{x})$ is just sufficient to meet interest payments on the oustanding consols.

If $g_T=G$, the bonds, held by the government, have value, and the consols held by the consumer do not. The government collects interest on its bonds, sells all of these bonds back to the consumer, and in addition issues (non-contingent) consols with a constant coupon payment of $\bar{a}$ each period. All of these revenues are used to help finance the current expenditure of G.

In periods $T+1,T+2,\ldots$, the situation is as in Example 4, regardless of whether $g_T=0$ or $g_T=G$.

Example 5 corresponds to a situation where there is a probability of war at some specified date in the future. It illustrates the risk-spreading aspects of optimal fiscal policy under uncertainty. In effect, the government in period $T-1$ buys insurance from the private sector: it promises to pay (the premium) $\bar{a}$ in all subsequent periods with $g_t=0$, in return for a claim to receive a payment ('damages') in period T, if the (unlucky) event $g_T=G$ occurs.

Example 6. Let ${}_0b\equiv 0$, let $g_t=G>0$, $t=T,T+S,T+2S,\ldots$, where $0\leqq T\leqq S$ (but $S\neq 0$), and let $g_t=0$, otherwise. From (2.9), the optimal allocation has the form $(c_t,x_t)=(\hat{c},\hat{x})$, $t=T,T+S,T+2S,\ldots$, and $(c_t,x_t)=(\bar{c},\bar{x})$, otherwise. Consequently, from (2.5) it follows that the tax rate also takes on two values,

$\hat{\tau}$ and $\bar{\tau}$, in war and peacetime years respectively. As in Example 4, tax revenue is positive during peacetime years, and indeterminate during wartime years. Thus, debt issues are as follows.

In each period $t=0,1,\ldots,T-1$, the government runs a surplus, which it uses to buy bonds issued by the private sector. In period T, the expenditure g_T is met by selling these bonds, possibly levying a tax on current labor income, and issuing new bonds. In periods $t=T+1,T+2,\ldots,S-1$, the government again runs a surplus, which is used to pay interest on and gradually to redeem the oustanding bonds. From (3.5) we see that λ_t is cyclic, with a cycle length of S periods. Thus, at $t=S$ the national debt is zero, and the cycle begins again.

Example 6 corresponds to perfectly foreseen, cyclic wars, with a cycle length of $S>0$ periods, where a war occurs $T\leqq S$ periods into each cycle. It is obvious from Example 5 that with any regular, cyclic expenditure pattern the budget will be balanced over the expenditure cycle.

Example 7. Let ${}_0b\equiv 0$ and $g_0=G>0$. If $g_t=G$, then $g_{t+1}=G$ with probability α, and $g_{t+1}=0$ with probability $1-\alpha$. If $g_t=0$, then $g_{t+1}=0$. As in the previous example, it follows from (2.9) that the optimal allocation has the form $(c_t,x_t)=(\hat{c},\hat{x})$ if $g_t=G$, and $(c_t,x_t)=(\bar{c},\bar{x})$ if $g_t=0$, all t, so that the tax rate takes on the values $\hat{\tau}$ and $\bar{\tau}$ during wartime and peacetime years respectively, with net tax revenue positive during peacetime years and indeterminate during wartime years. Let $\hat{a}$ and $\bar{a}$ denote the corresponding values for a_t.

Using (3.5), we can see how the war is financed. First, suppose that the war is still continuing in period $t>0$. From (3.5) and (3.4) it follows that if $g_t=G$, then $\lambda_t=\bar{\lambda}=\lambda_0$, and ${}_tb\equiv 0$. On the other hand, suppose that the war has ended by period $t>0$. From (3.5) and (3.4), it follows that if $g_t=0$, then $\lambda_t=\hat{\lambda}\neq\bar{\lambda}$, and ${}_tb=(1-\bar{\lambda}/\hat{\lambda})\bar{a}$. Consequently, the debt issues are as follows. While the war is in progress, it is financed at least in part through the issue of 'contingent bonds'. These bonds become consols, with constant coupon payment $\bar{a}$, if the war ends in the following period. If the war continues they become valueless. After the war ends, net tax revenue in each period is just sufficient to cover the current interest on the outstanding consols.

Example 7 corresponds to a war of unknown duration.

Example 8. Let ${}_0b\equiv 0$, and let $\{g_t\}$ be a sequence of independently and identically distributed random variables. From (2.17) it follows that the optimal allocation in period t, in state g^t, is a stationary function of g_t, so that the optimal allocation can be written as

$$(c_t(g^t),x_t(g^t))=(\gamma(g_t),\xi(g_t)),\qquad t=0,1,2,\ldots,\quad \text{all } g^t.$$

with corresponding values $a_t(g^t)=\alpha(g_t)$ for coupon payments on the optimal bond, and $\theta_t(g^t)=\Theta(g_t)$ for the optimal tax rate. It follows, then, using (3.5) and the fact that $\{g_t\}$ is i.i.d., that we can also write $\lambda_t(g^t)=\Lambda(g_t)$. Hence from (3.4), the quantity $(1-\Lambda(g_0)/\Lambda(g_t))$ of the government security outstanding in period t, in state g^t, depends only on g_0 and g_t. In particular, note that if $g_t=g_0$, then $(1-\lambda_0/\lambda_t)=0$, and there are no bonds outstanding.

Hence, debt restructurings occur as follows. In period t, given g_t, the government finds that its predecessor has left it with an obligation to pay $(1-\Lambda(g_0)/\Lambda(g_t))\alpha(g_t)$ units of goods in the current period and contingent obligations to pay $(1-\Lambda(g_0)/\Lambda(g_t))\alpha(G)$ units of goods in period s if the event $g_s=G$ occurs, for all $s>t$. Note that the obligation in any period $s>t$ is, at this point, contingent only on the realization of g_s.

Exactly the same statement must hold in period $t+1$, for every possible value of g_{t+1}. To ensure that this is the case, the government in period t must arrange that its end-of-period debt obligations are as follows:

(i) Contingent obligations to pay $(1-\Lambda(g_0)/\Lambda(G))\alpha(G)$ units of goods next period if $g_{t+1}=G$, all G.
(ii) Contingent obligations to pay $(1-\Lambda(g_0)/\Lambda(G))\alpha(G')$ units of goods in period s if the joint event $[g_{t+1}=G$ and $g_s=G']$ occurs, all G, G', all $s>t$.[5]

Example 9. Let ${}_0b\equiv 0$, and let $\{g_t\}$ be a stationary Markov process. The arguments and conclusions are exactly as in Example 8.[6]

The examples discussed in this section have not been chosen at random, but rather to illustrate some substantively important aspects of fiscal policy in practice. The shocks g_t that drive our system are government consumption *relative to* the ability of the economy to produce. In an economy like the United States, the main source of variation in g_t, so interpreted, are wars, brief and infrequent but economically very large when they occur, and business fluctuations, generally much smaller in magnitude but occurring more or less continuously. Examples 4–7 are designed to illustrate the main

[5]If U is quadratic, then $\Lambda(G)$ is a monotone increasing function. Thus, under the optimal policy, inherited (contingent) debt obligations are smaller conditional on higher current values for government consumption. This highlights the insurance aspect of optimal debt arrangements in the presence of uncertainty. Outstanding debt obligations are smaller in states with high current government consumption, where any current tax revenue is needed to help finance current government consumption, and excessively high tax rates are to be avoided — work must be encouraged to produce the relatively large quantity of goods c_t+g_t. In states with low current expenditure, taxes are used to repay previously incurred debt, or to build up a surplus.

[6]If $\{g_t\}$ is a Markov process, the monotonicity of the function Λ, discussed in footnote 5, can be expected only if the higher current levels of government consumption make higher levels in the following period, in some sense, more likely.

qualitative aspects of the public finance of wars. Examples 8 and 9, and their special case Example 3, attempt to capture more 'normal' situations.

Of the general lessons one can draw from these examples, three seem to us to be the most important. The first is simply built into the formulation at the outset: budget balance, in some *average* sense, is not something one can argue over in welfare-economic terms. If debt is taken seriously as a binding *real* commitment, then fiscal policies that involve occasional deficits necessarily involve offsetting surpluses at other dates. Thus in all of our examples with erratic government spending, good times are associated with budget surpluses.

Second, our examples illustrate once again the applicability of Ramsey's optimal taxation theory to dynamic situations, as articulated by Pigou (1947) and more recently by Kydland and Prescott (1980) and Barro (1979). In the face of erratic government expenditures, the role of debt issues and retirements is to smooth tax distortions over time, and it is clear that no general, welfare-economic case can be developed for budget balance on a *continuous* basis. Such a case (and nothing in our purely qualitative treatment suggests that it would be a weak one) would have to be based on the 'smoothness' of g_t (Example 3), and on some quantitative argument to the effect that an assumption of perfect smoothness is a useful approximation in some circumstances. Since it is easy to think of situations (Example 4) in which such an approximation would be a very bad one, it is clear that (as seems to be universally recognized) any welfare-improving commitment to budget balance will have to involve 'escape clauses' for exceptional (high g_t) situations.

Third, as is evident from all of the stochastic examples, the contingent-claim character of public debt is not in any sense an incidental feature of an optimal policy. Example 5 makes the insurance character of optimum debt issues clear, as does Example 7, in which a war-financing debt is repeatedly cancelled as long as the war continues, and is paid off only when the war ends. This feature is an entirely novel one in normative analysis of fiscal policy, to the point where even those most sceptical about the efficacy of actual government policy may be led to wonder why governments forego gains in everyone's welfare by issuing only debt that purports to be a *certain* claim on future goods.

Historically, however, nominally denominated debt has been anything but a certain claim on goods, and large-scale debt issues, typically associated with wars, have traditionally been associated with simultaneous and subsequent inflations that have, in effect, converted nominal debt into contingent claims on goods. Perhaps this centuries-old practice may be interpreted as a crude approximation to the kind of debt policies we have found to be optimal. Verifying this would involve going beyond the observation that war debts tend to be inflated away, in part, to establishing that the size of the inflation-induced 'default' on war debt bears some relation to the unanticipated size of the war. Example 7 states this issue about as baldly as it can be stated, but it can hardly be said to resolve it.

4. A monetary model

In this section, money, in the form of currency, is introduced into the economy studied in sections 2 and 3. We will first describe and motivate the specific way this will be carried out, paralleling as closely as possible the development of section 2. We consider two kinds of consumption goods, c_{1t} and c_{2t}, in addition to leisure x_t and government consumption g_t, all related by the technology

$$c_{1t}+c_{2t}+x_t+g_t \leqq 1, \qquad t=0,1,2,.., \quad \text{all } g^t, \tag{4.1}$$

where, as above, $\{g_t\}$ follows a stochastic process. Preferences are

$$E\left\{\sum_{t=0}^{\infty} \beta^t U(c_{1t},c_{2t},x_t)\right\}, \tag{4.2}$$

the expectation in (4.2) being taken with respect to the conditional distribution F_1^{∞} of the event $g_1^{\infty}=(g_1,g_2,\ldots)$, g_0 given.

The distinction between the two types of consumption, c_{1t} and c_{2t}, has to do with available payments arrangements, which we take to be as follows. The first good, c_{1t} ('cash goods'), can be purchased only with fiat currency previously accumulated. The second, c_{2t} ('credit goods'), can be paid for with labor income contemporaneously accrued. To clarify this distinction, consider the following trading scenario [taken in part from Lucas (1980)].

Think of a typical household as consisting of a worker–shopper pair, with one partner engaged each period in producing goods for sale and the other in travelling from store to store, purchasing a variety of consumption goods [all produced under the constant-returns technology (4.1)]. At some stores the shopper is known to the producer, who is willing to sell on trade-credit, the bill to be paid at the beginning of the next period. The total amount purchased on this basis, c_{2t}, we call 'credit goods'. At other stores the shopper is unknown to the seller, and any purchase must be paid for at once in currency. [Presumably the fact that the shopper is 'unknown' to the seller arises because there are resource costs involved in making oneself and one's credit-worthiness 'known' to someone else, but we do not pursue this here. See Prescott (1982).] Purchases made on this basis, c_{1t}, we call 'cash goods'. By postulating a current period utility function $U(c_{1t},c_{2t},x_t)$ with a diminishing marginal rate of substitution between cash goods and credit goods, we are assuming that only a limited range of goods is available on a credit basis, so that adding the option to substitute cash goods as well increases utility.

Although one might think of identifying cash and credit goods with observable consumption categories (food, clothing, and so on), we do not wish to do so here. On the contrary, think of one household's credit goods as being another's cash goods just as one can run up a tab at one's own neighborhood bar or grocery but not at others, or as it is worthwhile to establish credit in department stores in the city where one lives, but not in others. This is simply a matter of interpretation, since we offer no analysis of trade credit here, but it will matter in what follows that the 'inflation tax' is not interchangeable with an ordinary excise tax on some specific consumption category.

The *timing* of trading is important and we adopt the following conventions. At the beginning of period t, the shock g_t is realized and known to all. All agents, government included, convene in a centralized securities market. After outstanding debts are cleared, agents trade whatever securities (including currency) they choose. With this trading concluded, shoppers and producers disperse. Shoppers run down their cash holdings and accumulate bills. Producers accumulate cash and issue bills. These activities, together with arrangements entered into in securities trading, determine the household's consumption and leisure mix this period and the circumstances in which it begins the next period.

As in sections 2 and 3, a resource allocation $\{(c_{1t}, c_{2t}, x_t)\}_{t=0}^{\infty}$ is a sequence of contingent claims, the tth term of which is a function of the history g^t of shocks through that date. Price sequences are elements of the same space, as will be various securities to be specified in a moment. To develop the budget constraints faced by a household as of $t=0$, we use the prices $\{(q_t, p_t)\}$, where $q_t(g^t)$ is the dollar price at time 0 of a dollar at time t, contingent on the history g^t (so that, in particular, $q_0=1$), and where $p_t(g^t)$ is the current dollar price at time t of a unit of either type of goods at time t, contingent on g^t. Here 'at time t' means, more precisely, at the time of the 'morning' securities market in period t. Hence the price, in dollars at time 0, of a unit of cash goods in t, is $q_t(g^t)p_t(g^t)$, since the dollars must be acquired in the securities market held *prior* to (on the same day as) the goods purchase. The price at $t=0$ of a unit of credit goods in t is $q_{t+1}(g^{t+1})p(g^t)$, since bills are paid the day *after* the sale and consumption of such goods.

We imagine the household at $t=0$ as holding securities of two kinds: contingent claims $\{{}_0B_t\}$ to dollars at times $t=0,1,\ldots,$ priced at $\{q_t\}$, and contingent claims $\{{}_0b_{2t}\}$ to credit goods at times $t=0,1,\ldots,$ priced at $\{q_{t+1}p_t\}$ to coincide with the timing of payments for such goods. This set of securities is not comprehensive, as households might also wish to trade claims $\{{}_0b_{1t}\}$ to cash goods at times $t=0,1,\ldots.$ If such securities were available, however, they could be used by agents to circumvent the use of currency altogether, converting the system directly into the two-good barter economy studied at the end of section 2. This would conflict with our interpretation of cash goods as being anonymously purchased in spot markets only. To maintain the monetary interpretation of the model, then, direct claims to cash goods in 'real' terms will be ruled out.

The household's opportunity set, given prices and initial securities holdings, will then be described in two statements. One, describing options available in the centralized securities market, states that the dollar value of expenditures for all purposes is no greater than the dollar value of receipts from all sources. The other, describing options in decentralized cash goods markets, states that cash goods can only be purchased with currency.

The first of these constraints reads

$$\begin{aligned} &\int q_1 \mathrm{d}g_1[p_0c_{10} - M_0 + p_0c_{20} - p_0(1-\tau_0)(1-x_0) - p_{0\,0}b_{20}] \\ &\quad + \sum_{t=1}^{\infty} \iint q_{t+1}\mathrm{d}g_{t+1}[p_tc_{1t} - M_t + p_tc_{2t} - p_t(1-\tau_t)(1-x_t) - p_{t\,0}b_{2t}]\mathrm{d}g_1^t \\ &\quad + [M_0 - B_0] + \sum_{t=1}^{\infty} \int q_t[M_t - {}_0B_t]\mathrm{d}g_1^t \leqq 0, \end{aligned} \tag{4.3}$$

where $M_t \geqq 0$ denotes wealth held in the form of currency at the close of securities trading in period t. The first terms of (4.3) collect receipts and payments due at the beginning of period $t+1$, for $t=0,1,2,\ldots$, including unspent currency carried over from t, priced accordingly at q_{t+1}. The second terms collect returns on dollar-denominated securities in t less the amount held in currency. Since (4.3) contains terms of the form $[q_t(g^t) - \int q_{t+1}(g^{t+1})\,\mathrm{d}g_{t+1}]M_t(g^t)$, the budget constraint will be binding if and only if

$$q_t(g^{t+1}) - \int q_{t+1}(g^{t+1})\mathrm{d}g_{t+1} \geqq 0, \qquad t=0,1,2,\ldots, \quad \text{all } g^t. \tag{4.4}$$

If (4.4) is violated for any g^t the consumer can make arbitrarily large profits by holding arbitrarily large quantities of cash in state g^t. Thus, we will assume that (4.4) holds, or that the nominal interest rate is always non-negative.

Since currency must cover spending on cash goods, the second constraint is

$$p_tc_{1t} - M_t \leqq 0, \qquad t=0,1,2,\ldots, \quad \text{all } g^t. \tag{4.5}$$

This is simply the 'Clower constraint' proposed in Clower (1967), but applied to a subset of consumption goods only. Notice that if the function V is defined by $V(c_{1t}, c_{1t} + c_{2t}, x_t) \equiv U(c_{1t}, c_{2t}, x_t)$, and if (4.5) is always binding, current period utility is given by $U(M_t/P_t, c_{2t}, x_t) = V(M_t/P_t, c_{1t} + c_{2t}, x_t)$. So defined, V is the current period utility function used by Sidrauski (1967a, b), and by Turnovsky and Brock (1980). Hence, the imposition of a Clower constraint is not an alternative to Sidrauski's way of formulating the demand for money, but in fact is closely related to it.

The consumer's problem is then to maximize (4.2), subject to (4.3) and (4.5), given initial securities holdings $\{({}_0B_t, {}_0b_{2t})\}$, prices $\{(p_t, q_t)\}$ and tax rates $\{\tau_t\}$. Letting γ be the multiplier associated with (4.3), and letting $\rho_t(g^t)$ be the multiplier associated with (4.5) in state g^t, the first-order conditions for this problem are (4.3), (4.5) and

$$\beta^t U_1(c_{1t}, c_{2t}, x_t) f^t(g^t|g_0) - \gamma p_t \int q_{t+1} \mathrm{d}g_{t+1} - \rho_t p_t = 0, \tag{4.6}$$

$$\beta^t U_2(c_{1t}, c_{2t}, x_t) f^t(g^t|g_0) - \gamma p_t \int q_{t+1} \mathrm{d}g_{t+1} = 0, \tag{4.7}$$

$$\beta^t U_x(c_{1t}, c_{2t}, x_t) f^t(g^t|g_0) - \gamma p_t \int q_{t+1} \mathrm{d}g_{t+1}(1-\tau_t) = 0, \tag{4.8}$$

$$\gamma\left[\int q_{t+1} \mathrm{d}g_{t+1} - q_t\right] + \rho_t = 0, \qquad t = 0, 1, 2, \ldots, \text{ all } g^t \tag{4.9}$$

assuming, as we will, that c_{1t}, c_{2t}, x_t, and M_t are all strictly positive.

From (4.9) we see that if $\int q_{t+1} \mathrm{d}g_{t+1} - q_t < 0$, then $\rho_t > 0$, implying that (4.5) holds with equality. If $\int q_{t+1} \mathrm{d}g_{t+1} - q_t = 0$, then $\rho_t = 0$. In this case M_t is indeterminate within the constraint imposed by (4.5) (the consumer is indifferent between holding securities and excess cash), and we will assume that (4.5) holds with equality. Bearing in mind that any equilibrium obtained under this hypothesis must satisfy (4.4), (4.3) and (4.5) can be combined to give

$$\begin{aligned} 0 = &\int q_1 \mathrm{d}g_{t+1} p_0[(c_{20} - {}_0b_{20}) - (1-\tau_0)(1-x_0)] + p_0[c_{10} - {}_0B_0/p_0] \\ &+ \sum_{t=1}^{\infty} \int \Big(\int q_{t+1} \mathrm{d}g_{t+1} p_t[(c_{2t} - {}_0b_{2t}) - (1-\tau_t)(1-x_t)] \\ &+ q_t p_t[c_{1t} - {}_0B_t/p_t]\Big) \mathrm{d}g_1^t. \end{aligned} \tag{4.10}$$

Define ${}_0b_{1t} = {}_0B_t/p_t$ (so that ${}_0b_{1t}$ is dollar-denominated debt in 'real' terms). Then multiplying (4.10) through by γ and using (4.5)–(4.8) one obtains

$$\sum_{t=0}^{\infty} \beta^t \int [c_{1t} - {}_0b_{1t}, c_{2t} - {}_0b_{2t}, x_t - 1] \begin{bmatrix} U_1 \\ U_2 \\ U_x \end{bmatrix} \mathrm{d}F^t(g^t|g_0) = 0, \tag{4.11}$$

Note that (4.11) and the analogous condition (2.16) for the two-good barter economy studied in section 2 are formally identical. It is exactly this parallel that earlier writers have exploited in attempting to analyze the 'inflation tax' through analogy with the theory of excise taxes in barter systems. In the absence of both outstanding debt and government expenditures, efficiency would be attained [cf. (4.6)–(4.8)] if both the labor income tax rate τ_t and the multiplier ρ_t associated with the liquidity constraint (4.5) were set identically equal to zero. From (4.9), the latter requires $\int q_{t+1} \mathrm{d}g_{t+1} = q_t$, or a nominal interest rate identically zero, brought about by a deflation induced by continuous withdrawals of money from circulation. This is the conclusion Friedman (1969) reached, for the same reasons, but its implementation evidently depends critically on the availability of a non-distorting tax via which currency can be withdrawn.

If, as in Phelps (1973), Calvo (1978) or this paper, non-distorting taxes are assumed to be unavailable and if there are positive government obligations, then the formula (4.11) calls for taxing the two goods c_{1t} and c_{2t}, at rates that depend in Ramsey-like fashion on their relative demand elasticities. Here an income tax τ_t amounts to taxing both goods at the same rate, while an increase of the inflation tax from its 'optimum', zero-nominal-interest-rate level amounts to increasing the tax on cash goods, *relative to* credit goods. This leads to an important qualification to the analogy between (4.11) and (2.16): Since nominal interest rates cannot be negative in this monetary economy, cash goods can feasibly be taxed at a higher rate than credit goods but *not* at a lower one, whatever the relative demand elasticities may be. It leads as well to a substantial difference with Phelps's (1973) argument that 'liquidity' should be viewed as an *additional* good, with a presumption that an efficient tax program involves a positive inflation tax. In our framework, 'liquidity' (currency balances) is not a good, but rather the *means* to the acquisition of a subset of ordinary consumption goods. If one wishes to tax this subset at a higher rate than goods generally, the inflation tax is a means for doing so, but a positive interest-elasticity of money demand is clearly not sufficient to make this case.

Whatever the usefulness of these parallels between barter and monetary economies, all share a serious weakness once the issue of time consistency is raised. In the barter economy, we took the government at time 0 to be inheriting sequences, $\{({}_0b_{1t}, {}_0b_{2t})\}_{t=0}^{\infty}$ of binding *real* debt obligations, and to be choosing current excise tax rates, $(\theta_{10}, \theta_{20})$, and a restructuring of the debt, $\{({}_1b_{1t}, {}_1b_{2t})\}_{t=1}^{\infty}$. In the monetary economy, the time 0 government inherits real debt obligations $\{{}_0b_{2t}\}$ and nominal debt obligations $\{{}_0B_t\}$; it chooses the current tax rate τ_0 and, via an open market operation, the money supply M_0 in circulation when time 0 goods trading begins. The fact that (4.11) and (2.16) are formally identical is thus misleading, since $\{{}_0b_{1t}\}$ in (2.16) is a binding obligation, while $\{{}_0b_{1t}\}$ in (4.11) is not. The ability to choose M_0 indirectly gives the time 0 government the ability to affect the initial price level p_0 and all future price levels as well. From (4.10), one can see how this power is optimally used.

If the net value of initial nominal assets is positive [at any given equilibrium pattern $\{q_t\}$ of interest factors], welfare is improved by *any* increase in M_0 and p_0, since any increase reduces the real value of these assets and reduces the need to resort to the distorting tax on labor income to redeem the debt. Hence the optimal price level is 'infinite'. If the net value of initial nominal assets is negative, the best monetary policy is the one that sets the value of these assets equal to the net value of all current and future government spending. In this way, *all* distorting taxation can be avoided. In the first situation, an optimal policy with commitment does not exist. In the second, an optimal policy exists and it is time-consistent (since fully efficient allocations always are so), but it is one based on circumstances bearing little resemblance to those faced by any actual government.

The remaining possibility, and the only one, we think, of potential practical interest, is the situation in which ${}_0B_t \equiv 0$, so that initially there are no outstanding nominal obligations of any kind. In this situation, the ability to manipulate nominal prices through open market operations offers no immediate possibilities for welfare gains. The setting of the initial price level is simply a matter of normalization. For this particular case, then, we will first look for an optimal policy with full commitment by the government at $t=0$, specifying the tax rates, money supplies, and nominal and real debt issues needed to implement this policy, and the equilibrium prices and interest rates associated with it. With this done, we will try to determine the weakest possible commitments under which the optimal policy might be carried out in a time-consistent way.

An allocation $\{(c_{1t}, c_{2t}, x_t)\}$ satisfying (4.11) with ${}_0b_{1t} \equiv 0$ can be implemented by suitable choices of tax rates and money supplies $\{(\tau_t, M_t)\}$. From (4.7) and (4.8), the required taxes are

$$1-\tau_t = U_x(c_t, x_t)/U_2(c_t, x_t), \qquad t=0,1,2,\dots, \quad \text{all } g^t. \tag{4.12}$$

From (4.6), (4.7) and (4.9), the required nominal interest factors satisfy

$$\int q_{t+1}\,\mathrm{d}g_{t+1} = q_t(U_2(c_t, x_t)/U_1(c_t, x_t)), \qquad t=0,1,2,\dots, \quad \text{all } g^t. \tag{4.13}$$

From (4.4) and (4.6)

$$p_t q_t \gamma = \beta^t U_1(c_t, x_t) f^t(g_1^t \,|\, g_0)$$

so that with $q_0 \equiv 1$,

$$q_t(g^t) = \beta^t \frac{U_1(c_t, x_t) f^t(g_1^t \,|\, g_0)}{U_1(c_0, x_0)} \frac{p_0}{p_t}. \tag{4.14}$$

Thus given a contingent path for prices $\{p_t\}$, (4.14) determines nominal, state-contingent interest rates.

Use the notation $f_{t+1}(g_{t+1}|g_0^t)$ for the density of g_{t+1} conditional on the history g_0^t. Then $f^{t+1}(g_1^{t+1}|g_0)=f_{t+1}(g_{t+1}|g_0^t)f^t(g_1^t|g_0)$, so integrating (4.14) dated $t+1$ with respect to g_{t+1} gives

$$\int q_{t+1}\,\mathrm{d}g_{t+1}=\beta^{t+1}f^t(g_1^t|g_0)[p_0/U_1(c_0,x_0)]$$

$$\times\int\frac{U_1(c_{t+1},x_{t+1})}{p_{t+1}}f_{t+1}(g_{t+1}\,|\,g_0^t)\,\mathrm{d}g_{t+1}.$$

Inserting the equation above and (4.14) into (4.13), we find that

$$\frac{U_2(c_t,x_t)}{p_t}=\beta\int\frac{U_1(c_{t+1},x_{t+1})}{p_{t+1}}f_{t+1}(g_{t+1}\,|\,g_0^t)\,\mathrm{d}g_{t+1}. \tag{4.15}$$

Now any allocation $\{(c_t,x_t)\}$ satisfying (4.11) may be implemented as follows. Tax rates $\{\tau_t\}$ are uniquely given in (4.12). There is much more latitude, however, in the choice of monetary policy. First, note that for any price path $\{p_t\}$ satisfying (4.15), $\{q_t\}$ as given in (4.14) satisfies (4.13). Given *any* such price path, it may be implemented by the associated monetary policy

$$M_t=p_tc_{1t}.$$

Clearly, there are many such price paths and associated monetary policies, and all are feasible provided (4.4) is not violated. Since all are associated with the same resource allocation, all are equivalent from a welfare point of view.

Since the constraint (4.4) must also hold in equilibrium, (4.13) implies that in addition to satisfying (4.11), feasible allocations must also satisfy

$$U_2(c_t,x_t)-U_1(c_t,x_t)\leqq 0,\qquad t=0,1,2,\ldots,\quad \text{all } g^t. \tag{4.16}$$

The optimal open-loop allocation for the monetary economy, then, is found by choosing $\{(c_{1t},c_{2t},x_t)\}$ to maximize (4.2) subject to (4.1), (4.11) and (4.16).

The first-order conditions for this problem, consolidated in such a way as to parallel condition (2.17) for the n-good barter system, are

$$(1+\lambda_0)U'+\lambda_0U''\begin{bmatrix}c_t-{}_0b_t\\ x_t-1\end{bmatrix}-\mu_{0t}\mathbf{1}-v_t\begin{bmatrix}U_{21}-U_{11}\\ U_{22}-U_{12}\\ U_{2x}-U_{1x}\end{bmatrix}=\mathbf{0},\quad\text{and} \tag{4.17}$$

$$v_t(U_2-U_1)=0,\qquad t=0,1,2,\ldots,\quad \text{all } g^t, \tag{4.18}$$

where $v_t f'_1 \beta^t$ is the non-negative multiplier associated with the constraint (4.16), and λ_0 is the multiplier associated with (4.11). If (4.16) is never binding, so that $v_t = 0$ for all t, g^t, then (4.17) reduces to (2.17), and the case under consideration reduces exactly to the two-good barter system of section 2.

Let $\{(c_{1t}, c_{2t}, x_t)\}_{t=0}^{\infty}$ be a solution of (4.1), (4.11), and (4.16)–(4.18). Let $\{\tau_t\}_{t=0}^{\infty}$ be given by (4.12), let $\{p_t\}_{t=0}^{\infty}$ be any price path satisfying (4.15), let $\{q_t\}_{t=0}^{\infty}$ satisfy (4.14), and $\{M_t\}_{t=0}^{\infty}$ to be given by $M_t = p_t c_{1t}$. Under what conditions might this optimal policy be time-consistent?

It is clear from the debt-restructuring formulas of section 2 that, in general, the debt issues needed to enforce time-consistency in a two-good economy will involve claims to *both* of the two goods. In the present monetary interpretation of this two-good economy, issuing claims to cash goods, b_{1t}, can be done only through the issue of dollar-denominated assets B_t. Yet we have seen above that any dollar-denominated assets inherited by those governments will be inflated away by them if they are acting in a welfare-maximizing way. Anticipating this, no one would buy such debt at a positive price. There is, in short, no hope that an optimal policy will be time-consistent (will be a closed loop equilibrium policy) with fiscal and monetary policy both determined in an unrestricted, period-by-period way, except under special and uninteresting circumstances.

What is needed for time-consistency in the monetary economy is that nominal debt *always* represent a binding *real* commitment. Since $b_{1t} = B_t/p_t$, a nominal commitment B_t can be equivalent to a real commitment b_{1t} *only* if there is also a commitment to follow a specific price path p_t. Thus the following scenario is the closest imitation the monetary economy can provide to the optimal, time-consistent solution in the barter economy.

Let the initial government take office with no nominal assets in the hands of the public. Let it calculate the optimal (open loop) allocation, as above, along with the corresponding tax and monetary policies and associated prices, with initial money arbitrarily chosen. Let this government choose the initial tax rate τ_0, announce future taxes $\{\tau_t\}_{t=1}^{\infty}$, and *precommit* future monetary policy to enforce *some* price path satisfying (4.15). Finally, let this initial government restructure the initial real debt $\{{}_0b_{2t}\}_{t=0}^{\infty}$ into a new pattern $\{({}_1B_t, {}_1b_{2t})\}_{t=1}^{\infty}$, of nominal and real debt. Subsequent governments will have full control over future tax rates and over restructurings of debt of both kinds, but no ability to alter the original precommitment on future price level behavior.

Under this scenario, the time-consistency of the optimal policy (in the restricted sense of the paragraph above) follows as a corollary of the time-consistency proof of section 2. The government taking office at $t = 1$, in deciding whether to execute the tax policy announced by its predecessor at $t = 0$, is faced with a severely restricted set of available actions as compared to the government in section 2 (one tax rate to choose instead of two) but the optimal choice of section 2 is in the restricted set. Hence it will be chosen, and time-consistency follows.

Notice that this argument does not go through if the government precommits itself to a monetary path $\{M_t\}$ instead of a price path $\{p_t\}$. For a given money supply, one sees from the condition $M_t = p_t c_{1t}$ that different consumption levels c_{1t} of cash goods will induce different price level behavior, and the income tax rate τ_t can clearly affect c_{1t}. Hence a monetary rule would leave open the possibility of using tax policies to alter the degree to which nominal debt commitments B_t are binding, a possibility that will clearly change the marginal conditions on which our proof of time-consistency in section 2 was based.

The mechanics by which a price precommitment of the sort used above would be carried out are exactly the same as in any monetary standard: the government announces (and backs up, if needed) its willingness to exchange any quantities of currency for goods at the state-contingent prices $\{p_t\}$. The amount of currency actually set into circulation is then fully 'demand determined'. In equilibrium, this announcement does not necessitate any government holdings of commodity 'stockpiles' (which is lucky, since we have assumed that all goods are perishable!).

5. Remarks on scope and applicability

By considering a closed system with identical consumers, we have abstracted from consideration of conflict between a 'creditor class' and a 'debtor class' a conflict on which historical discussion of national debt policy has been almost exclusively focused. We also denied ourselves the use of the 'small country' device of treating national debt by analogy with the theory of individual debt in a competitive world. We have, in short, restricted attention to situations in which the half-truth 'We only owe it to ourselves' becomes a whole-truth. These abstractions evidently exclude some issues of interest, but they clearly heighten the difficulty of the time-consistency problem. Thus our conclusions as to the necessity and efficacy of government debt obligations being binding in a real sense on successor governments have nothing to do either with maintaining a reputation that impresses outside creditors or with limiting the options open to 'bad' (in the sense of having different objectives from our own) future governments.

The exclusion of capital goods from the model is central, for reasons that are easy enough to see from section 4. In the model of that section, outstanding nominal assets should, from a welfare-maximizing point of view, be taxed away via an immediate inflation in a kind of 'capital levy'. This emerged as a new possibility when money was introduced in section 4 *only* because capital had been excluded from the barter analysis of section 2. Had the taxation of previously accumulated capital been an option in section 2, then it would optimally have been exercised and we would have needed to face this capital levy issue two sections earlier.

Clearly this limitation on the scope of our results is important, and it would be a total misreading of our paper to take its main lesson to be that the time-consistency problem is easy to solve in barter systems and hard only when money is introduced. We stepped around questions about capital not because they are minor or easy, but because they are difficult and basically different from the issues we wanted to address The main difficulty, as Chamley (1982) observes, is that direct capital levies can be imitated — to perfection, under some circumstances — by combinations of taxes and subsidies that look, superficially, like taxes on current and future decisions only, so that it is hard to devise simple ways to rule them out. However this question may ultimately be resolved, it seems to us different from the ones we have addressed, and it is likely that our main conclusions will be little altered by such a resolution. At present, this opinion is clearly conjecture only.

The assumption that government consumption is determined, perhaps stochastically, by 'nature' (and not by public choice) seems, for our purposes, innocuous. It may be that a deeper look at this issue will reveal a relationship between this assumption and our presumption that while a society can commit itself to an infinite sequence of contingent claim bond payments, it cannot commit itself to a sequence of tax rates, contingent on precisely the same events. Within our formalism, this distinction is inexplicable: the two forms of commitment are describable mathematically as elements of precisely the same space. Why should one represent a practical possibility, the other an impossibility? Yet the idea, that while a government may issue binding debts, the nature of the taxes needed to repay them should be a matter decided by the citizens subject to the tax at the time this decision is taken, is one that we accept almost without question in policy discussion. If a rationale for this presumption is found, it may well be connected to the public choice aspects of government consumption, or to the idea that if our successors are to be free to *choose* to do more or less through government than we anticipate we would do, given their circumstances, then they cannot very well be committed in advance to a pattern of taxes prescribed by us. It seems clear enough that the model utilized here is not well designed to make progress on this class of questions.

Finally, our emphasis on calculating *exact* welfare-maximizing policies may be misleading in a sense worth commenting on. Clearly, a policy or policy rule that is optimal in a theoretical model that is an approximation to reality, can only be approximately optimal applied in reality. This observation suggests that in practice one would probably seek price commitments or bond commitments that are simple and also serviceable approximations to optimal, and perhaps quite complicated, contingent claim commitments, as calculated above. The models we have used, particularly the quadratic examples of appendix A, are well suited to assessing the 'welfare costs' of arbitrary policies relative to optimal ones, and formulae for expected-utility differences of this type could be obtained. At the qualitative, illustrative level at which we are working, we did not find such formulae very revealing, and so did not inflict them on the reader, but with a quantitatively more serious model this line would be well worth developing. Certainly the idea of trying to write bond contracts or set monetary standards in a way that is optimal under *all possible* realizations of shocks would not (even if one knew what that meant) be of any practical interest.

6. Conclusions

This paper has been concerned with the structure and time-consistency of optimal tax policy in two multiperiod economies: a pure barter system and a monetary economy, both without capital goods. In each case, the government had to choose a method of financing an exogenous stochastic sequence of government expenditures. Current consumption goods and a complete set of contingent claim securities were assumed to be traded in each period.

In section 2, we showed that the optimal tax policy is time-consistent, provided that fully binding debt of a sufficiently rich maturity and risk structure can be issued, and that the optimal debt policy is unique. A single debt instrument, a kind of contingent-claim consol, was shown to be the only form of debt needed to enforce time-consistency. In section 3, the optimal tax policy was characterized under a variety of assumptions about the behavior of government consumption. From the examples with stochastic government demand, it was clear that the option to issue state-contingent government debt is important: tax policies that are optimal under uncertainty have an essential 'insurance' aspect to them.

In section 4 money, in the form of currency, was introduced via a transactions demand, along with nominally-denominated debt. The analogy between the monetary economy and a two-good barter system permitted us to apply the analysis of section 2. Our conclusion paralleled familiar results on the 'optimal inflation tax' or 'optimal quantity of money'. However, the analogy with the barter system broke down when time-consistency was considered. The ability to use discretionary monetary policy to levy an 'inflation tax' cannot be disciplined by binding debt issues in the way that ordinary excise taxation can be. Time-consistency can be achieved only if monetary policy is pre-set to maintain a specified path of nominal prices. Somewhat surprisingly, this same effect cannot be achieved through a pre-set path for the quantity of money, since the interaction of fiscal and monetary policy permits tax policies to alter the effects on prices of any given monetary policy.

In a general way, our findings serve to reinforce Kydland and Prescott's (1977) arguments to the effect that some form of institutional commitment is essential for the implementation of fiscal and monetary policies that have desirable effects under the usual welfare-economic criteria. We have tried to make some progress on what seems to us the central task of discovering exactly which forms of commitment are sufficient and what functions they serve.

Appendix A

This appendix describes the calculation of the optimal fiscal policy for the one good model studied in sections 2 and 3, for the case of a quadratic utility function $U(c,x)$. We provide necessary and sufficient conditions for the existence of a unique optimal policy for this case, and give exact formulae for some of the relationships alluded to in the text.

Let $(\bar{c},\bar{x})$ maximize $U(c,x)$, subject to $c+x\leqq 1$, and let δ denote the common value of $U_c(\bar{c},\bar{x})$ and $U_x(\bar{c},\bar{x})$. Expanding the marginal utilities of consumption and leisure about $(\bar{c},\bar{x})$ and using (2.1) to eliminate x, we have

$$U_c(c,x)=\delta+(U_{cc}-U_{cx})(c-\bar{c})-U_{cx}g, \tag{A.1}$$

$$U_x(c,x)=\delta+(U_{cx}-U_{xx})(c-\bar{c})-U_{xx}g. \tag{A.2}$$

In this quadratic case, the derivatives U_{cc}, U_{cx} and U_{xx} are constant and (A.1) and (A.2) are exact. We proceed with the construction of an optimal allocation, as sketched in section 2.

For notational convenience, define

$$\Delta=-[U_{cc}-2U_{cx}+U_{xx}],\quad \text{and} \tag{A.3}$$

$$v=-\Delta^{-1}(U_{xx}-U_{cx}). \tag{A.4}$$

Since U is concave, $\Delta>0$, and since both goods are normal (non-inferior), $0<v<1$. Note that v is the derivative of leisure demand with respect to income y in the problem: maximize $U(c,x)$, subject to $c+x\leqq y$, and $1-v$ is the derivative of goods demand. In this notation the solution c_t to the first order conditions (2.1) and (2.9) is given explicitly by

$$c_t=\frac{1+\lambda}{1+2\lambda}\bar{c}-vg_t+\frac{\lambda}{1+2\lambda}(1-v)_0b_t \tag{A.5}$$

(where the subscript on λ_0 has been dropped). This is the only solution, and it is a local maximum. It is convenient to let $\mu \equiv (1+2\lambda)^{-1}\lambda$, so that (A.5) reads

$$c_t = (1-\mu)\bar{c} - vg_t + \mu(1-v)_0 b_t. \tag{A.6}$$

Then the constraint (2.8) reads

$$\sum_{t=0}^{\infty} \beta^t E\{\mu(1-\mu)\Delta[\bar{c}-(1-v)_0 b_t]^2 - \delta(_0 b_t + g_t) - \alpha g_t(_0 b_t + g_t)\} = 0, \tag{A.7}$$

where $E\{\ \}$ denotes an expected value taken with respect to F, given g_0 and α is defined by

$$\alpha = \Delta^{-1}(U_{xx}U_{cc} - U_{cx}^2), \tag{A.8}$$

which is positive for a risk-averse consumer. Then solving (A.7) for μ gives

$$\mu(1-\mu) = \left[\Delta \sum_{t=0}^{\infty} \beta^t E\{[\bar{c}-(1-v)_0 b_t]^2\}\right]^{-1} \sum_{t=0}^{\infty} \beta^t E\{\delta(_0 b_t + g_t) + \alpha g_t(_0 b_t + g_t)\}. \tag{A.9}$$

Provided $g_t \geqq 0$ and $_0 b_t < \bar{c}/(1-v)$, the right-hand side of (A.9) is non-negative. It is also increasing in each term of $_0 b_t$ and g_t. If the right-hand side of (A.9) exceeds 1/4, no real value of μ satisfies (A.9). This is what was meant in section 2 by the looser statement that no optimal policy will exist if $_0 b$ and g are 'too large'. If, as assumed here, this expression is less than 1/4, (A.9) has two solutions for μ, one in the interval $(-\infty, \frac{1}{2})$, the other in $(\frac{1}{2}, 1)$. The smaller of these two roots corresponds to the welfare-maximizing solution of interest to us. Notice that if $_0 b_t$ is sufficiently negative, $\mu < 0$ is possible. Thus, the questions of the existence and uniqueness of an optimal allocation are easily resolved in this specific case.

With $\mu \in (0, \frac{1}{2})$, both μ and $1-\mu$ are positive. Thus from (A.6), under an optimal fiscal policy c_t declines as g_t increases, but less than one-for-one unless the income elasticity of leisure demand is zero ($v=0$); c_t increases with debt obligations b_t, unless the income elasticity of consumption demand is zero ($v=1$). When the government budget constraint (A.9) is not binding, $\mu = 0$ and $c_t = \bar{c}$.

In Examples 4–8 of section 3, initial debt commitments $_0 b$ were taken to be zero. Under this circumstance, in this quadratic case, the bond coupon formula (3.4) becomes

$$_t b_s = \left(1 - \frac{\lambda_0}{\lambda_t}\right)\frac{\bar{c}}{1-v}. \tag{A.10}$$

Since the right-hand side of (A.10) does not vary with s, only consols are ever issued. The formula (A.9) for μ reduces to

$$\mu(1-\mu)=(1-\beta)(\Delta\bar{c}^2)^{-1}\sum_{t=0}^{\infty}\beta^t E\{\delta g_t+\alpha g_t^2\} \tag{A.11}$$

and the optimum consumption formula (A.6) becomes simply

$$c_t=(1-\mu)\bar{c}-vg_t. \tag{A.12}$$

It is instructive to apply (A.10)–(A.12) to Examples 4–8, but this exercise is left to the interested reader.

Appendix B

For a broad class of optimal policy problems, if an optimal policy with commitment is time-consistent (as defined in section 2), then that policy corresponds to a set of subgame perfect Nash equilibrium strategies for an appropriately specified game.

A typical policy game can be specified as follows. The set of players is $0, 1, 2, \ldots$, where player t is the policy-maker in period t. Let Y_t denote the set of possible states of the system in period t, and assume that player t observes (at least) the state $y_t \in Y_t$. Let $A_t(y_t)$, denote the set of actions available to player t if the state is y_t. A strategy for player t is a function σ_t such that $\sigma_t(y_t) \in A_t(y_t)$, all $y_t \in Y_t$. Let S_t denote the set of all such functions, and let S_t be the strategy space for player t. (Mixed strategies could readily be incorporated without altering the rest of the argument.) Define $\sigma_t^\infty \equiv (\sigma_t, \sigma_{t+1}, \ldots)$, and $S_t^\infty \equiv (S_t, S_{t+1}, \ldots)$, all t.

The law of motion for the system is as follows. Let $M_{t+1}(B|(y_t, a_t))$, for all $B \subseteq Y_{t+1}$, all $y_t \in Y_t$, all $a_t \in A_t(y_t)$, be the conditional probability that the state in period $t+1$ is in the subset B of Y_{t+1}, i.e., that $y_{t+1} \in B \subseteq Y_{t+1}$, given that the state in period t is y_t, and the (feasible) action $a_t \in A_t(y_t)$ was taken.

Next, we must specify a payoff function for each of the players. The payoff for player t will depend only on the current state, y_t, his own strategy σ_t [which specifies his action $\sigma_t(y_t)$], and the strategies of his successors, σ_{t+1}^∞, (which specify, together with the law of motion, a joint probability distribution over future states and actions). Let $\pi_t(\sigma_t^\infty, y_t)$ denote player t's payoff function.

Then under the definition in section 2, a set of strategies (policy) σ_0^∞ is *time consistent* if

$$\pi_t(\sigma_t^\infty, y_t) \geq \pi_t(\hat{\sigma}_t^\infty, y_t), \qquad \text{for all } \hat{\sigma}_t^\infty \in S_t^\infty, \quad y_t \in Y_t, \quad t \in T,$$

while a set of strategies σ_0^∞ is a *subgame perfect Nash equilibrium* if

$$\pi_t(\sigma_t^\infty, y_t) \geqq \pi_t(\hat{\sigma}_t, \sigma_{t+1}^\infty, y_t), \quad \text{for all } \hat{\sigma}_t \in S_t, \quad y_t \in Y_t, \quad t \in T.$$

Clearly the former condition implies the latter.

For the game in section 2, the state in period t is described by the outstanding debt and the sequence of government consumption to date, $y_t = ({}_t b, g^t)$; the actions available to player t are the choice of a tax rate and debt restructuring, $a_t = (\tau_t, {}_{t+1}b)$; a strategy σ_t for player t maps states $({}_t b, g^t)$ into current policy $(\tau_t, {}_{t+1}b)$; the law of motion is

$$M_{t+1}((B_b, B_g) | ({}_t b, g^t), (\tau_t, {}_{t+1}b)) = \int_{B_g} \mathrm{d}F^{t+1}(g^{t+1} | g^t) \quad \text{if} \quad {}_{t+1}b \in B_b,$$

$$= 0, \quad \text{otherwise},$$

where $({}_{t+1}b, g^{t+1}) \in (B_b, B_g)$; and the payoff function for player t is

$$\pi_t(\sigma_t^\infty, ({}_t b, g^t)) = E\left[\sum_{s=t}^{\infty} \beta^{s-t} U(c_s, x_s)\right],$$

where $\{(c_s, x_s)\}_{s=t}^\infty$ is the (perfect foresight) equilibrium allocation resulting from the initial state $({}_t b, g^t)$, when the governments in periods $t, t+1, \ldots,$ choose policies according to $\sigma_t, \sigma_{t+1}, \ldots$.

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