Sensitivity Analysis for Golosov, Hassler, Krusell, and Tsyvinski (2013): "Optimal Taxes on Fossil Fuel in General Equilibrium"

Lint Barrage*

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1 Overview

This document studies the sensitivity of the optimal carbon tax formulation derived by Golosov, Hassler, Krusell, and Tsyvinski (2013) ("GHKT"). GHKT show that, under certain assumptions, the optimal carbon tax-GDP ratio can be solved for in closed form, and does not depend on the paths of future output, consumption, and technological change. These assumptions include logarithmic preferences and full depreciation of capital over the course of a decade.

This document relaxes these assumptions and explores the numerical sensitivity of the optimal carbon tax-GDP ratio to the structure of preferences, depreciation, and technological progress. It further proposes a slightly modified version of GHKT's central optimal carbon tax formulation that approximates the optimal carbon tax in the case of non-logarithmic constant elasticity utility and non-zero long-run productivity growth.

The remainder of this note is structured as follows. Section 2 reviews the planner's problem as presented in GHKT (2013), and then describes our numerical implementation. Section 3 outlines the sensitivity analyses considered, and presents the main quantitative results. Section 4 proposes a modification of GHTK's formula that approximates the optimal carbon tax in the case that preferences are not logarithmic and productivity growth is positive. Finally, Appendix Section 5 compares the numerical model's benchmark case results with those from the true, infinite-horizon problem as presented in GHTK (2013).

*University of Maryland, Department of Agricultural and Resource Economics. Email: lbarrage@arec.umd.edu

2 Model

2.1 Recap of GHKT Model

2.1.1 GHKT General Model

This section reviews the theoretical framework presented by GHKT. As we abstract from uncertainty throughout this document, we present a simplified, deterministic version of the GHKT model. A global representative household has preferences over consumption C_t:

$$\sum_{t=0}^{\infty} \beta^t U(C_t) \tag{1}$$

There are I production sectors: $I-1$ intermediate energy good producing sectors, indexed by $i = 1, ...I$, and one final consumption-investment good sector, indexed by $i = 0$. The final goods resource constraint is given by:

$$C_t + K_{t+1} = Y_t + (1-\delta)K_t \tag{2}$$

where K_t denotes the (aggregate) capital stock. Final good output Y_t is produced from technology $F_{0,t}$:

$$Y_t = F_{0,t}(K_{0,t}, N_{0,t}, \mathbf{E}_{0,t}, S_t) \tag{3}$$

where $N_{0,t}$ is labor allocated to the final goods sector, and $E_{0,t} = (E_{0,1,t}, E_{0,2,t}, ..., E_{0,I,t})$ denotes a vector of energy inputs. Output further depends on the state of the climate, S_t, taken here as the atmospheric carbon stock. All climate change impacts are thus represented as production damages.

Energy input i is produced from technology:

$$E_{i,t} = F_{i,t}(K_{i,t}, N_{i,t}, \mathbf{E}_{i,t} R_{i,t}) \geq 0 \tag{4}$$

For energy resources in finite supply - such as petroleum - $R_{i,t}$ denotes the stock of resource i still left at the beginning of period t. The resource stock evolves according to:

$$R_{i,t+1} = R_{i,t} - E_{i,t} \geq 0 \tag{5}$$

Factors are assumed to be perfectly mobile across sectors, implying that:

$$\sum_{i=0}^{I} K_{i,t} = K_t, \sum_{i=0}^{I} N_{i,t} = N_t, \text{ and} \sum_{i=0}^{I} E_{i,j,t} = E_{j,t} \tag{6}$$

Lastly, energy inputs $i = 1, ..., I_g - 1$ are assumed to be carbon-based, whereas inputs $i = I_g, ...I$ are "green" and not associated with carbon emissions. All energy inputs are given in terms of carbon content (equivalent). Atmospheric carbon concentrations S_t are thus a function $\widetilde{S}_t$ of carbon-based energy inputs dating back to the start of industrialization at time $-T$:

$$S_t = \widetilde{S}_t \left(\sum_{i=1}^{I_g - 1} E_{i,-T}, E^f_{-T+1}, ..., E^f_t \right) \tag{7}$$

where $E^f_t \equiv \sum_{i=1}^{I_g-1} E_{i,t}$ denotes the sum of fossil fuel inputs in tons of carbon.

The government's problem is to maximize (1) subject to (2), (3), (4), (5), (6), and (7). As demonstrated by GHKT, comparison of the planner's first-order conditions with the decentralized equilibrium conditions governing the behavior of firms and households suggests that the optimal allocation is implemented by a Pigouvian carbon tax. This tax is equal to the marginal externality damages of carbon emissions from energy input i, $\Lambda^s_{i,t}$:

$$\Lambda^s_{i,t} \equiv \sum_{j=0}^{\infty} \beta^j \frac{U'(C_{t+j})}{U'(C_t)} \frac{\partial F_{0,t+j}}{\partial S_{t+j}} \frac{\partial S_{t+j}}{\partial E_{i,t}} \tag{8}$$

Finally, since energy inputs $E_{i,t}$ are all recorded in tons of carbon, it is moreover the case that:

$$\begin{aligned} \frac{\partial S_t}{\partial E_{i,t}} &= \frac{\partial S_t}{\partial E_{j,t}} \forall i, j, \in \{1, ...I_g - 1\} \\ &\Rightarrow \Lambda^s_{i,t} = \Lambda^s_{j,t} = \Lambda^s_t \end{aligned}$$

2.1.2 GHKT Benchmark Assumptions

GHKT derive a closed-form expression for the optimal carbon tax-GDP ratio by imposing only the following assumptions:

Assumption 1 $U(C_t) = \ln(C_t)$

Assumption 2 $F_{0,t}(K_{0,t}, N_{0,t}, \mathbf{E}_{0,t}, S_t) = (1 - D_t(S_t))\widetilde{F}_{0,t}(K_{0,t}, N_{0,t}, \mathbf{E}_{0,t})$ with

$$1 - D_t(S_t) = \exp(-\gamma_t(S_t - \overline{S}))$$

and where $\overline{S}$ denotes pre-industrial carbon concentrations.

Assumption 3 The function $\widetilde{S}_t$ is linear with the following depreciation structure:

$$S_t - \overline{S} = \sum_{s=0}^{t+T} (1 - d_s)\, E_{t-s}^f$$

and $d_s \in [0, 1]$ for all s.

Assumption 4 Full depreciation: $\delta = 1$

Given assumptions (1)-(4), GHKT demonstrate that the optimal carbon tax is a simple formulation that depends only on discounting, the climate damage parameter γ_t, and the carbon depreciation structure:

$$\Lambda_t^s = Y_t \left[\sum_{j=0}^{\infty} \gamma_{t+j} (1 - d_j) \right] \tag{9}$$

GHKT's quantitative analysis parameterizes the carbon depreciation structure as follows:

$$1 - d_s = \phi_L + (1 - \phi_L)\phi_0 (1 - \phi)^s \tag{10}$$

where ϕ_L denotes the share of carbon emissions that remains permanently in the atmosphere, fraction $(1 - \phi_0)$ of emissions exit the atmosphere immediately (through absorption in the biosphere and upper ocean), and the remainder of emissions decays at geometric rate ϕ. Given (10), we finally arrive at the "Benchmark formulation" for the optimal carbon tax-GDP ratio:

$$\widehat{\Lambda}_t^s \equiv \frac{\Lambda_t^s}{Y_t} = \gamma_t \left(\frac{\phi_l}{1 - \beta} + \frac{(1 - \phi_l)\phi_0}{1 - (1 - \phi)\beta} \right) \tag{11}$$

The central objective of this note is to study the sensitivity of (11) to relaxing assumptions (1) and (4). In that case, $\widehat{\Lambda}_t^s$ depends also on the future paths of output and consumption (8). We thus also study the sensitivity of $\widehat{\Lambda}_t^s$ to general assumptions about future technological change in the more general environment without assumptions (1) and (4).

2.1.3 GHKT Benchmark Full Model

GHKT provide a full characterization and quantitative results for optimal carbon taxes and allocations for the following version of the general model outlined above. Note that assumptions (1)-(4) are maintained throughout.

Energy Sector: There are three energy sectors: oil, coal, and clean energy. Oil inputs, indexed by $i = 1$, are assumed to be in finite supply R_0. Oil extraction is assumed to be costless:

$$E_{1,t} = R_t - R_{t+1} \tag{12}$$

Coal and clean energy, indexed by $i = 2$ and $i = 3$, respectively, are produced using only labor inputs. Constraint (4) thus becomes:

$$E_{i,t} = A_{i,t} N_{i,t} \text{ for } i = 2, 3 \tag{13}$$

Final Goods Sector: The final goods production technology is assumed to be Cobb-Douglas:

$$Y_t = e^{-\gamma_t (S_t - \overline{S})} A_{0,t} K_t^{\alpha} N_{0,t}^{1-\alpha-v} E_t^{v} \tag{14}$$

Here, the energy composite E_t is given by:

$$E_t = \left(\kappa_1 E_{1,t}^{\rho} + \kappa_2 E_{2,t}^{\rho} + \kappa_3 E_{3,t}^{\rho}\right)^{1/\rho} \tag{15}$$

with $\sum_{i=1}^{3} \kappa_i = 1$.

Carbon Cycle: The history of carbon emissions prior to period zero is dealt with as follows. Stock S_1 denotes the carbon that remains in the atmosphere forever, whereas stock S_2 denotes depreciating atmospheric carbon. These and the total atmospheric carbon stock then evolve according to:

$$\begin{aligned} S_{1,t} &= S_{1,t-1} + \phi_l E_t^f \\ (S_{2,t} - \overline{S}) &= \phi(S_{2,t} - \overline{S}) + \phi_0 (1 - \phi_L) E_t^f \\ S_t &= S_{1,t} + S_{2,t} \end{aligned} \tag{16}$$

Given (12)-(16), GHKT analytically characterize and numerically solve for optimal allocations and energy input paths in particular.

Quantitative Implementation: GHKT solve for optimal allocations by combining the planner's optimality conditions from the *infinite* horizon problem (as discussed above) with the assumption that all oil is used up over the course of a *finite* time horizon T considered:

$$\sum_{t=0}^{T} E_{1,t} = R_0 \tag{17}$$

Since oil usage goes to zero as T approaches infinity, (17) should serve as decent approximation for sufficiently large values of $T < \infty$.

Another key feature of the Benchmark case that enables GHKT's algorithm is that the optimal carbon tax-GDP ratio $\widehat{\Lambda}_t^s$ is exogenous and constant given assumptions (1)-(4). That is, the formulation (11) captures the infinite-horizon present value of climate damages without the need to actually compute output or consumption over an infinite time horizon. However, this simplification no longer holds in the more general case without assumptions (1) and (4). In the more general case (8), one needs to know $\{Y_t\}_{t=0}^{\infty}$ and $\{C_t\}_{t=0}^{\infty}$ to compute the optimal carbon tax-GDP ratio $\widehat{\Lambda}_t^s$. The next section thus describes our numerical approximation to the planner's problem that we use to explore the sensitivity of $\widehat{\Lambda}_t^s$.

2.2 Numerical Model for Sensitivity Analysis

Our numerical model generally maintains the functional forms of the Benchmark GHKT model (12)-(16), with a few modifications as discussed below. Given the high number of state variables in the problem, we do not employ value function iteration. Instead, we construct a direct optimization program that seeks to approximate the planner's true, infinite-horizon problem as follows. First, the program directly optimizes over all allocations for $T < \infty$ periods. After period T, a continuation value V_T is computed as a function of the last direct optimization period's carbon stock S_T, capital stock K_T, savings rate θ_{T-1}, oil extraction rate Θ_{T-1}, and the shares of labor devoted to the production of coal, clean energy, and final output, respectively. As discussed below, this continuation value assumes that a balanced growth path is eventually reached.

We consider a constant elasticity formulation of preferences which nests the Benchmark case of logarithmic preferences when $\sigma = 1$. The planner's problem is thus:

$$\max_{\mathbf{X}} \sum_{t=0}^{T-1} \beta^t \left(\frac{C_t(\mathbf{X})^{1-\sigma} - 1}{1-\sigma} \right) + \beta^T V_T(\mathbf{X}) \tag{18}$$

where the vector of choice variables $\mathbf{X}$ is given by:

$$\mathbf{X} = [\{\theta_t\}_{t=0}^{T-1}, \{R_{t+1}\}_{t=0}^{T-1}, \{\pi_{2t}\}_{t=0}^{T}, \{\pi_{3t}\}_{t=0}^{T}] \tag{19}$$

Here, θ_t denotes the gross savings rate in period t, and π_{it} is the share of labor devoted to sector i at time t. For each guess of $\widehat{\mathbf{X}}$, the implied sequence of consumption $\{C_t(\widehat{\mathbf{X}})\}_{t=0}^{T}$ can be computed as described below, along with continuation value $V_T(\widehat{\mathbf{X}})$.

2.2.1 Bounds and Constraints

We impose the following lower and upper bounds on the choice variables in (19):

$$\begin{aligned}0 &\le \theta_t \le 1\\ 0 &\le R_{t+1} \le R_0\\ 0 &\le \pi_{2t} \le 1\\ 0 &\le \pi_{3t} \le 1\end{aligned}$$

For all $t = \{0, ...T\}$, we further impose a non-negativity constraint on consumption. For numerical optimization purposes, this constraint is actually implemented as requiring slightly positive consumption:

$$C_t > 0.00001$$

2.2.2 Objective Function: Computation of $\{C_t(\widehat{\mathbf{X}})\}_{t=0}^{T}$

This section describes how $\{C_t(\widehat{\mathbf{X}})\}_{t=0}^{T}$ is computed (within the objective function) for a given guess of the direct optimization choice variables (19).

Energy Inputs: For periods $t = \{0, ..., T-1\}$, total energy inputs E_t can be inferred by substituting oil stocks and labor shares into the energy production functions (12), (13), and (15):

$$E_t = \{\kappa_1 (R_t - R_{t+1})^\rho + \kappa_2 (A_{2t}\pi_{2t}N)^\rho + \kappa_3 (A_{3t}\pi_{3t}N)^\rho\}^{1/\rho} \tag{20}$$

To compute oil consumption during and after period T, we treat oil extraction rates in period $T-1$ as steady-state values that are continued thereafter. That is, define the period $T-1$ oil extraction rate Θ_{T-1} as the fraction of oil in the ground at the beginning of period $T-1$ that is extracted during period $T-1$:

$$\Theta_{T-1} \equiv \frac{E_{1,T-1}}{R_{T-1}} = \frac{R_{T-2} - R_{T-1}}{R_{T-1}}$$

Period T oil consumption and the oil stock at time $T+1$ are then given by:

$$\begin{aligned}R_{T+1} &= R_T \cdot (1 - \Theta_{T-1})\\ E_{1,T} &= \Theta_{T-1} \cdot R_T \end{aligned} \tag{21}$$

Note that this approach differs from the GHKT's Benchmark Numerical Model approximation that all oil is used up over the course of $T < \infty$ period (17). We should thus expect to see marginally different oil extraction paths when comparing this model's results with those of the GHKT Benchmark Numerical Model.

Given (21), along with π_{2T} and π_{3T} from $\widehat{\mathbf{X}}$, we can back out period T energy inputs, E_T:

$$E_T = \left\{\kappa_1 E_{1,T}^{\rho} + \kappa_2 \left(A_{2T}\pi_{2T}N\right)^{\rho} + \kappa_3 \left(A_{3T}\pi_{3T}N\right)^{\rho}\right\}^{1/\rho} \tag{22}$$

Carbon Emissions and Concentrations: The amounts of carbon-based fossil fuel inputs implied by $\widehat{\mathbf{X}}$ can be easily computed by substituting into the energy production functions (12) and (13), as applied in (20). In contrast to the standard GHKT model, however, we introduce a form of technological progress that reduces the emissions intensity of coal usage over time. Specifically, let ϑ_t denote the fraction of coal's carbon-equivalent energy content that ends up emitted from combustion at time t. Carbon emissions E_t^m for periods $t = \{0, ..., T\}$ can then be computed from $\widehat{\mathbf{X}}$ via:

$$E_t^f = (R_t - R_{t+1}) + \vartheta_t(A_{2t}\pi_{2t}N) \tag{23}$$

The introduction of ϑ_t is motivated by the need to assume a balanced growth path at some point in time. If ϑ_t goes to zero as t approaches infinity, then carbon emissions will go to zero as well, since oil usage is continually declining. In this setting, assuming stabilized carbon concentrations after time T should be an acceptable approximation to the true model for sufficiently large T.

Intuitively, declining emissions intensity ϑ_t can also be motivated as reflecting increasingly cost-competitive abatement possibilities. The seminal DICE climate-economy model (e.g.,. Nordhaus, 2010) assumes that the economy becomes slightly less carbon-intensive over time even without climate policy interventions, and that carbon emissions abatement costs likewise decrease over time due to technological progress. While the representation of energy inputs and carbon emissions is quite different in the DICE and GHKT models, we nonetheless argue that a gradually declining coal emissions intensity ϑ_t is broadly in line with similar concepts from the literature. As a first pass, we assume a logistic functional form for ϑ_t, with parameters a and b :

$$\vartheta_t = \frac{1}{1 + \exp(-(a + b(t))} \tag{24}$$

Figure 2.1 displays the ϑ_t, function over time for the parameters we maintain throughout this note.

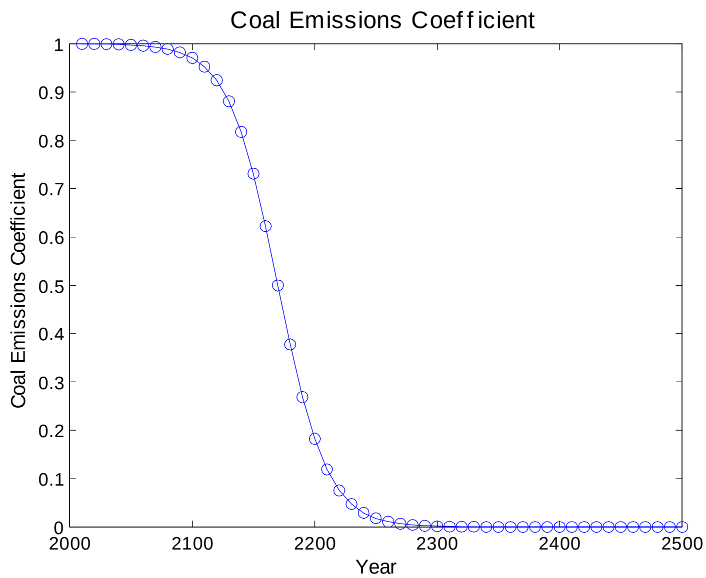


Figure 2.1

For our calibration, the emissions intensity of coal only begins to substantially decrease after the year 2100.

Output and Consumption: Finally, given $\{E_t\}_{t=0}^{T}$, $\{S_t\}_{t=0}^{T}$, and $\widehat{\mathbf{X}}$, we can compute output, consumption, and capital for periods $t = \{0, ..., T-1\}$ from production function (14) and the aggregate resource constraint (2):

$$Y_t = e^{-\gamma(S_t - \overline{S})} A_t K_t^{\alpha} \left\{(1 - \pi_{2,t} - \pi_{3,t})N\right\}^{1-\alpha-v} E_t^{v} \tag{25}$$

$$C_t = (1 - \theta_t) Y_t \tag{26}$$

$$K_{t+1} = \theta_t Y_t + (1 - \delta) K_t \tag{27}$$

For periods T and thereafter, we treat the savings rate in period $T-1$ as steady-state value that is subsequently maintained. For consumption in period $t = T$, we thus have that:

$$C_T = (1 - \theta_{T-1}) Y_T$$

2.2.3 Objective Function: Computation of $V_T(\widehat{\mathbf{X}})$

After period T, based on the values of K_T, S_T, R_T, and the 'steady-state' choice variables $\Theta_{T-1}, \theta_{T-1}, \pi_{2T,}$, and π_{3T}, the continuation value $V_T(\widehat{\mathbf{X}})$ is computed as follows.

First, we simulate the continuation of the economy for n periods after T. Specifically, for periods $T+j$, $j \in \{1, 2, ...n\}$, we have that oil continues to be extracted at rate Θ_{T-1} as in (21):

$$\begin{aligned} E_{1,T+j} &= \Theta_{T-1} \cdot [R_{T+j}] \\ &= \Theta_{T-1} \cdot \left[R_T(1-\Theta_{T-1})^j\right] \end{aligned}$$

Coal and clean energy inputs grow at the long-term rate of labor productivity growth, gZ :

$$\begin{aligned} E_{2,T+j} &= (A_{2,T+j})\pi_{2,T}N \\ &= (1+g_Z)^j E_{2,T} \end{aligned}$$

$$\begin{aligned} E_{3,T+j} &= (A_{3,T+j})\pi_{3,T}N \\ &= (1+g_Z)^j E_{3,T} \end{aligned}$$

Energy inputs continue to follow (20):

$$E_{T+j} = \left\{\kappa_1 \left(E_{1,T+j}\right)^\rho + \kappa_2 \left(E_{2,T+j}\right)^\rho + \kappa_3 \left(E_{3,T+j}\right)^\rho\right\}^{1/\rho} \tag{28}$$

For large enough T, coal emissions intensity ϑ_{T+j} will be close to zero, and oil usage $E_{1,T+j}$ should be low. We thus impose that carbon concentrations have reached their new steady-state value by period T :

$$S_{T+j} = S_T$$

Given (28), K_T, and $\widehat{\mathbf{X}}$, we can compute Y_{T+j}, K_{T+j}, and C_{T+j} analogously to (25)-(27):

$$Y_{T+j} = A_{T+j}(e^{-\gamma_T(S_T-\overline{S})})(K_{T+j}{}^\alpha)\left\{(1-\pi_{2T}-\pi_{3T})N\right\}^{1-\alpha-v} E_{T+j}^v)$$

$$K_{T+j} = (\theta_{T-1})Y_{T+j-1} + (1-\delta)K_{T+j-1}$$

and:

$$C_{T+j} = (1 - \theta_{T-1})Y_{T+j}$$

After period $T+n$, we assume that the economy has reached a balanced growth path, and that consumption grows at constant rate $(1 + g_{BGp})$:

$$C_{T+n+j} = (1 + g_{BGP})^j (C_{T+n+j})$$

Finally, the continuation value of the objective function is thus given by:

$$V_T(\widehat{\mathbf{X}}) = \sum_{j=0}^{T+n} \beta^{T+j} \left(\frac{C_{T+j}(\widehat{\mathbf{X}})^{1-\sigma} - 1}{1-\sigma} \right) + \beta^{T+n} \left(\frac{C_{T+n}(\widehat{\mathbf{X}})^{1-\sigma} - 1}{1-\sigma} \right) \left[\frac{1}{1 - \beta(1 + g_{BGP})^{1-\sigma}} \right] \quad (29)$$

3 Calibration and Results

3.1 Calibration

Table 1 provides GHKT's Benchmark quantitative analysis parameters as well as the alternative values considered in this sensitivity analysis.

Here, the "DICE" value for $gTFP_t$ represents the time-varying TFP growth rates to which the 2010 DICE Model (Nordhaus, 2010) is calibrated. This growth rate, gA_t^{NH}, is given by:

$$gA_t^{NH} = gA_0^{NH} \exp(-\gamma_0 \cdot t \cdot \exp(-\gamma_1 \cdot t))$$

where t is time in the number of years (i.e., t for the first period it $t = 10$), and where:

$$\begin{aligned} gA_0^{NH} &= 0.160023196685654 \\ \gamma_0 &= 0.00942588385340332 \\ \gamma_1 &= 0.00192375245926376 \end{aligned}$$

Here, $\gamma_0 \sim$ rate of decline in productivity growth rate (percent per year), $\gamma_1 \sim$ rate of decline of decline in productivity growth rate (percent per year), and $gA_0^{NH} \sim$ initial rate of productivity growth per decade. After period T, we impose that $gA_{T+j}^{NH} = gA_T^{NH} = 3.27\% \; \forall j$. That is, long-run TFP growth is assumed to remain at $\sim 0.32\%$ per year. Figure 3.1 depictes the annual TFP growth rate implied by gA_t^{NH} :

Parameter	Benchmark	Alt1	Alt2	Alt3
σ	1	1.5	2	0.5
$gTFP_t$ (% per year)	0.0%	1.3%	1.5%	$DICE$
δ (% per decade)	100%	65%		
β (annual)	.985	.990	.995	.999
gA_{2t}, gA_{3t} (% per year)	2.0%			
ρ	-0.058			
κ_1	0.5429			
κ_2	0.1015			
κ_3	0.3556			
$A_{2,0}$	7693			
$A_{3,0}$	1311			
R_0 (GtC)	253.8			
N	1			
ϕ	0.0228			
ϕ_L	0.2			
ϕ_0	0.393			
$\overline{S}$ (GtC)	581			
$S_{1,-1}$ (GtC)	103			
$S_{2,-1}$ (GtC)	699			
γ	0.000023793			
α	0.3			
v	0.04			
N (normalized)	1			

Table 1: Calibration Parameters

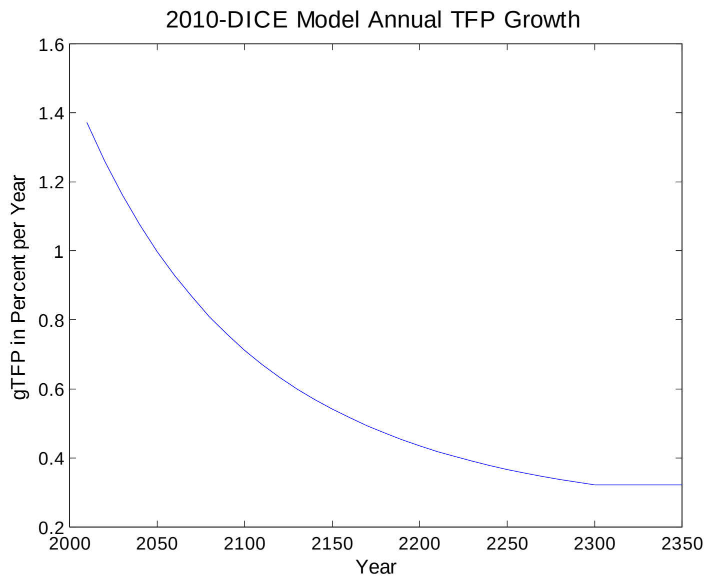


Figure 3.1

Several further parameters do not have a counterpart in the GHKT Benchmark case, and/or are unimportant for the computation of optimal carbon taxes and energy paths. These include:

Parameter	Benchmark	Alternative
a	8	
b	-0.05	
A_0	$17,887$	$16,640$
K_0 (US$ bil.)	$128,920$	$164,030$

A notable challenge in the calibration of initial final good sector productivity A_0 and the initial capital stock K_0 is the decision whether or not to re-calibrate these values when we change the assumed capital depreciation rate, δ. Our general approach to calibrating K_0 and A_0 is to match a representative *net* rate of return on capital of 5% per year (as in, e.g., the 2010 DICE Model, Nordhaus, 2010), corresponding to a net decadal return of $\widetilde{r} = 62.89\%$. That is, given world GDP in the calibration period $t = -1$ (calendar year 2009), we solve for K_0 via:

$$K_0 = \frac{\alpha(Y_{2009} \cdot 10)}{r} = \frac{\alpha(Y_{2009} \cdot 10)}{\widetilde{r} + \delta} \tag{30}$$

where $r = (\widetilde{r} + \delta)$ equals the gross return on capital, and Y_{2009} is *annual* GDP in the calibration year 2009. For a given net rate of return on capital, the main issue is thus that the decadal gross return r differs depending on whether we assume a decadal depreciation rate of $\delta = 1$, or $\delta = .65$ (corresponding to an annual depreciation rate of 10%). This question of re-calibration matters both because it determines how far the economy is from its balanced growth path capital-output ratio, and because GDP levels will grow more rapidly with higher initial TFP, which grows at an assumed, exogenous rate.[1] On the other hand, re-calibration requires changing multiple parameters at once, thus rendering the interpretation of differences in results across experiments more difficult. We deal with this issue by reporting results for both re-calibrated and non-recalibrated values when changing the depreciation rate.

3.2 Computation

The computation is performed in Matlab using the 'Active Set' algorithm in *fmincon*. The direct optimization time horizon is set to $T = 30$ periods $= 300$ years. The subsequent simulation horizon for the computation of the continuation value V_T is set to $n = 100$ periods $= 1000$ years. To maintain numerical precision, aggregate consumption is recorded in quadrillions of dollars for the evaluation in the objective function, an the convergence tolerance is set to $1 \cdot e^{-12}$.

3.3 Main Results

First, Figure 3.2 plots $\widehat{\Lambda}_t$ over time for the main cases considered with $\beta = .985$ in order to provide a broad sense for the order of magnitude of variations in $\widehat{\Lambda}_t$ observed:

[1] For a given K_0, we infer initial TFP based on:

$$A_0 = \frac{(Y_{2009} \cdot 10)}{e^{-\gamma(S_0 - \overline{S})} K_0^{\alpha} \left\{(1 - \pi_{2,0} - \pi_{3,0})N\right\}^{1-\alpha-v} E_0^{v}}$$

where $\pi_{2,0}$ and $\pi_{3,0}$ are normalized to zero to match the GHKT Benchmark calibration underlying the energy production technologies.

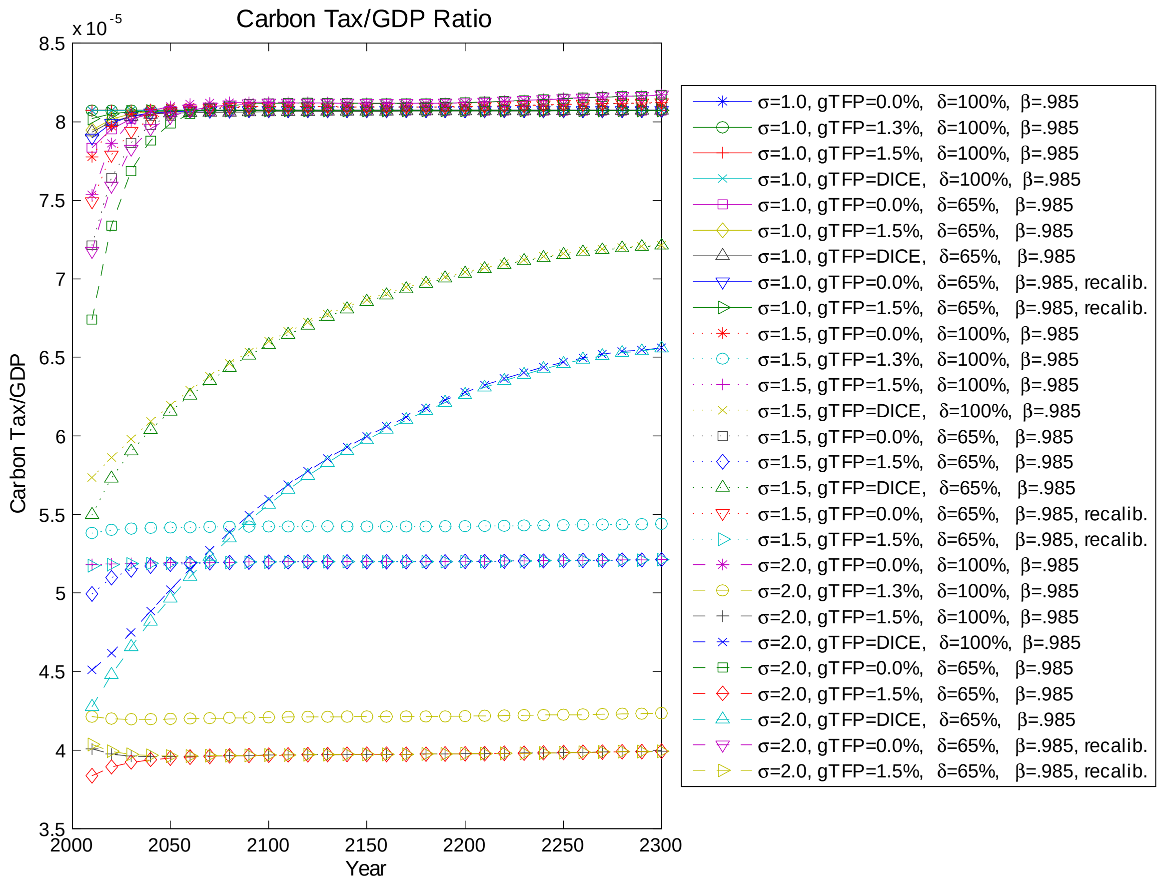


Figure 3.2

The GHKT benchmark case ($\sigma = 1$, $\delta = 100\%$) has a constant optimal carbon tax-GDP ratio of 8.07 $\times 10^{-5}$. The results in Figure 3.2 suggest that consideration of higher curvature in the utility function, coupled with positive TFP growth, can decrease the optimal carbon tax-GDP ratio by up to 50 percent in the case of ($\sigma = 2$, $gTFP = 1.5\%$). As discussed below in Section 4, with a slight tweak, GHKT's benchmark optimal carbon tax formulation (11) can predict these differences in $\widehat{\Lambda}_t$ well. Section 4 further demonstrates that adjusting β to approximately maintain the effective discount factor from the benchmark case when changing σ and $gTFP$ produces $\widehat{\Lambda}_t$ close to the benchmark as well. The main results in Figure 3.2 further suggest that the optimal carbon tax-GDP ratio is essentially constant in most cases considered. The main exception occurs when $\sigma > 1$ and with the time-varying TFP growth rates from the DICE model (Nordhaus, 2010). As discussed below, this is because agents' effective discount rate keeps on changing along with $gTFP$ in this case.

Next, in order to zoom in on the impacts of depreciation rates and productivity growth, Figures 3.3, 3.4, and 3.5 show $\widehat{\Lambda}_t$ over time for fixed combinations of the intertemporal preference parameters σ and β.

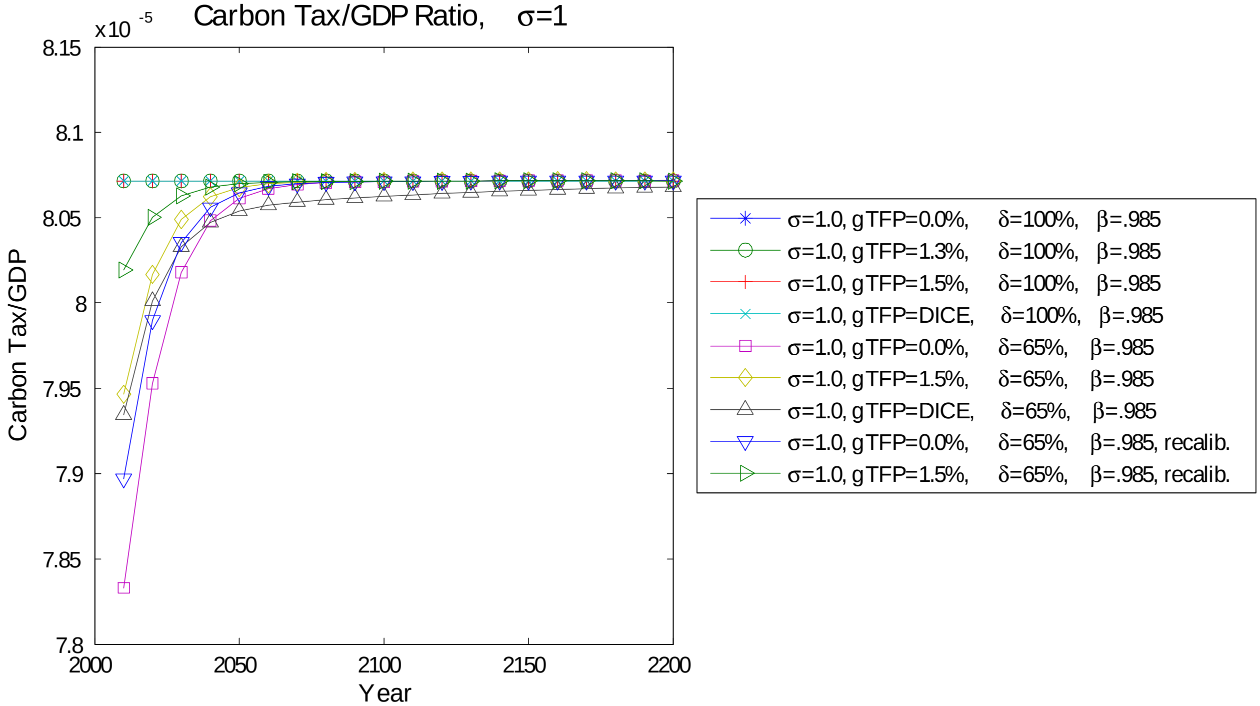


Figure 3.3

With logarithmic preferences ($\sigma = 1$), we see that differences in TFP growth rates do not affect the optimal carbon tax relative to GDP, as expected. Consideration of less-than-full depreciation introduces transitional dynamics which lead to a temporary deviation from the benchmark $\widehat{\Lambda}_t$. However, the impact of depreciation is quantitatively modest, and transitional dynamics are predicted to be fast.

Next, with more than logarithmic curvature in the representative agent's utility function, we find that higher TFP growth decreases the optimal carbon-GDP ratio. Time-varying TFP growth rates ($gTFP = DICE$) moreover lead to changes in $\widehat{\Lambda}_t$ over time. However, similar to the benchmark case, consideration of less-than-full depreciation has only a brief and quantitatively small impact on $\widehat{\Lambda}_t$.

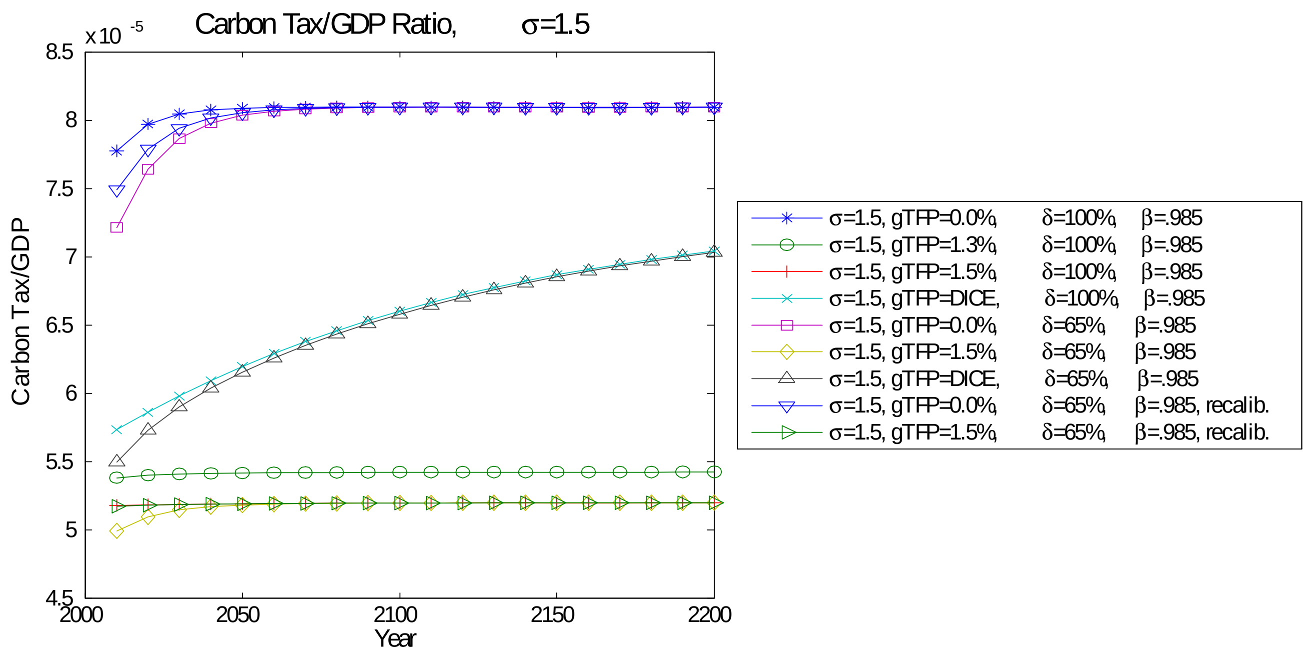


Figure 3.4

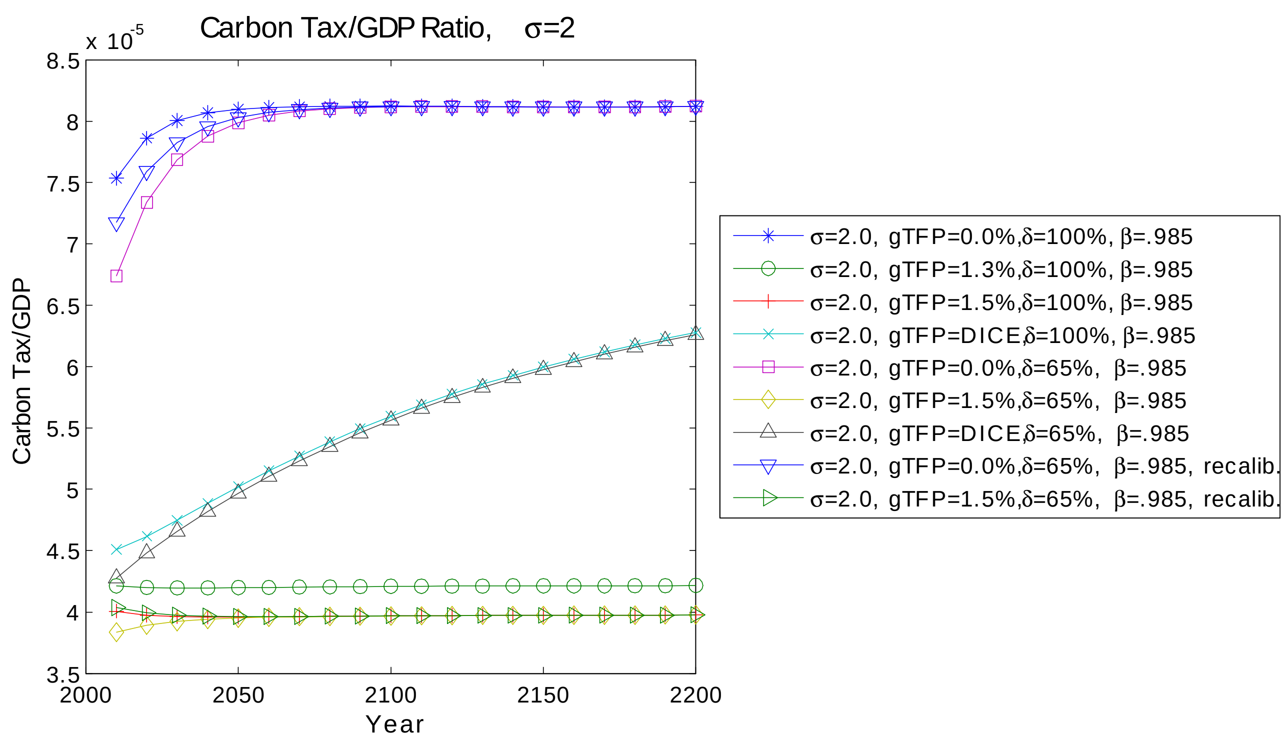


Figure 3.5

The previous results all focus on a pure rate of social time preference of 1.5% per year. Figure 3.6 displays the optimal carbon tax-GDP ratios in 2010 across different values of β :

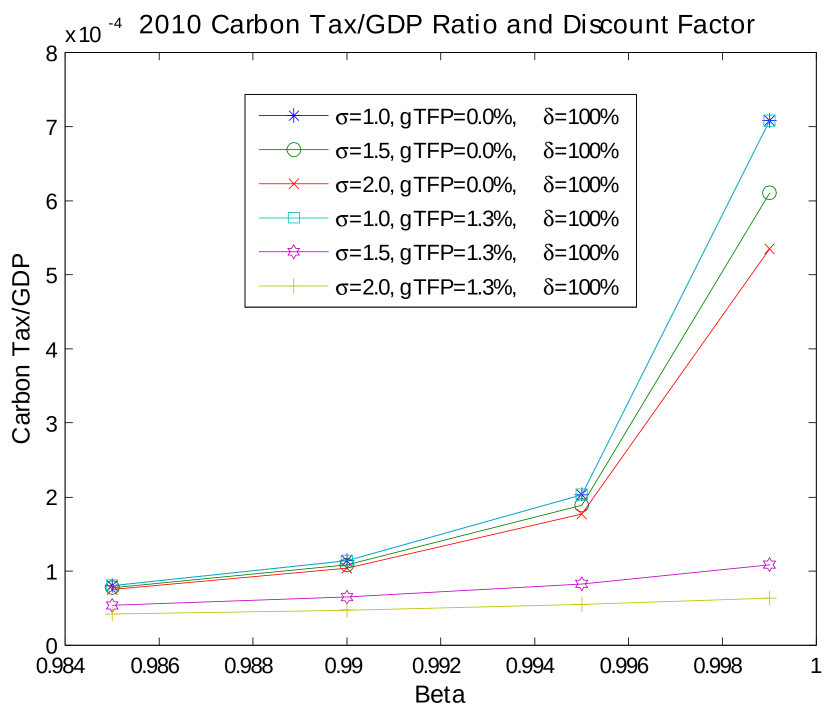


Figure 3.6

The results suggest that the optimal carbon tax-GDP ratio is most sensitive to the discount factor with logarithmic preferences. As seen above, changes in the TFP growth rate do not affect $\widehat{\Lambda}_t$ in this case. In contrast, with higher curvature in the utility function, TFP growth greatly diminishes the sensitivity of the optimal carbon tax-GDP ratio to changes in β.

Finally, Figure 3.7 shows changes in the optimal year 2010 carbon tax-GDP ratio as a function of σ, the inverse intertemporal elasticity of substitution.

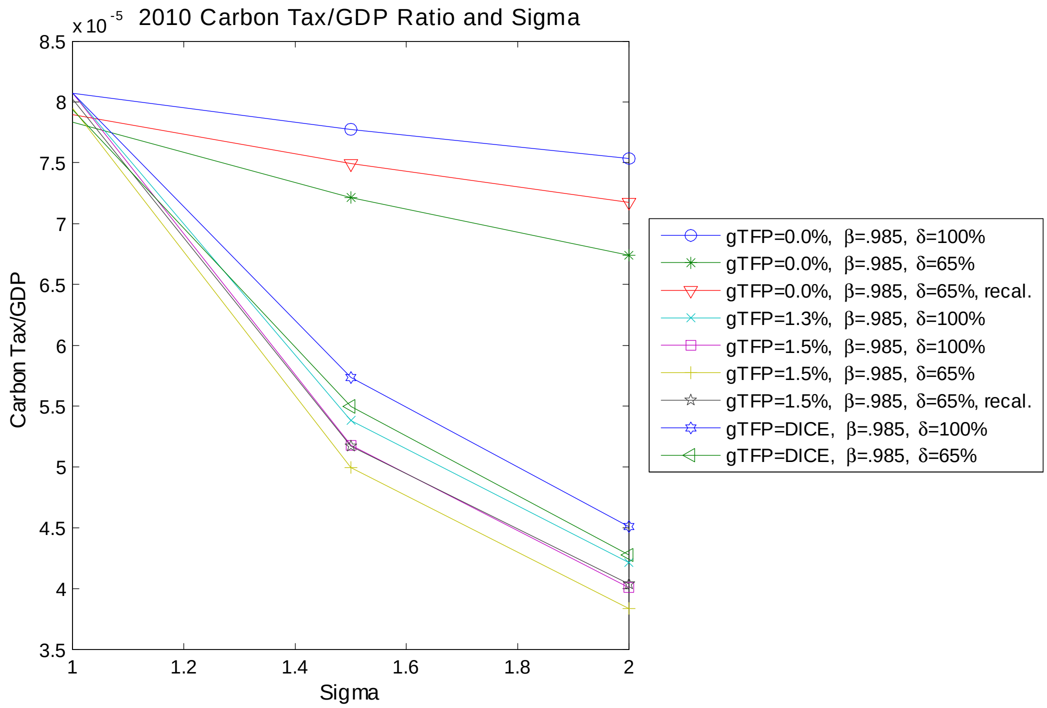


Figure 3.7

Without TFP growth, $\widehat{\Lambda}_t$ appears quite robust to changes in σ. However, with TFP growth, the optimal carbon tax-GDP ratio is decreasing in σ, as seen above.

3.4 Optimal Carbon Tax Levels

The analysis has thus far focused on the optimal carbon tax-GDP ratio. However, as GDP during the initial decade responds endogenously to changes in preferences and technological progress, it is also potentially interesting to consider changes in optimal carbon tax *levels* due to the parameter variations considered.

Figure 3.8 depicts optimal carbon tax levels in USD ($2000) per metric ton carbon over the course of the next 100 years (for a pure rate of social time preference $\beta = .985$):

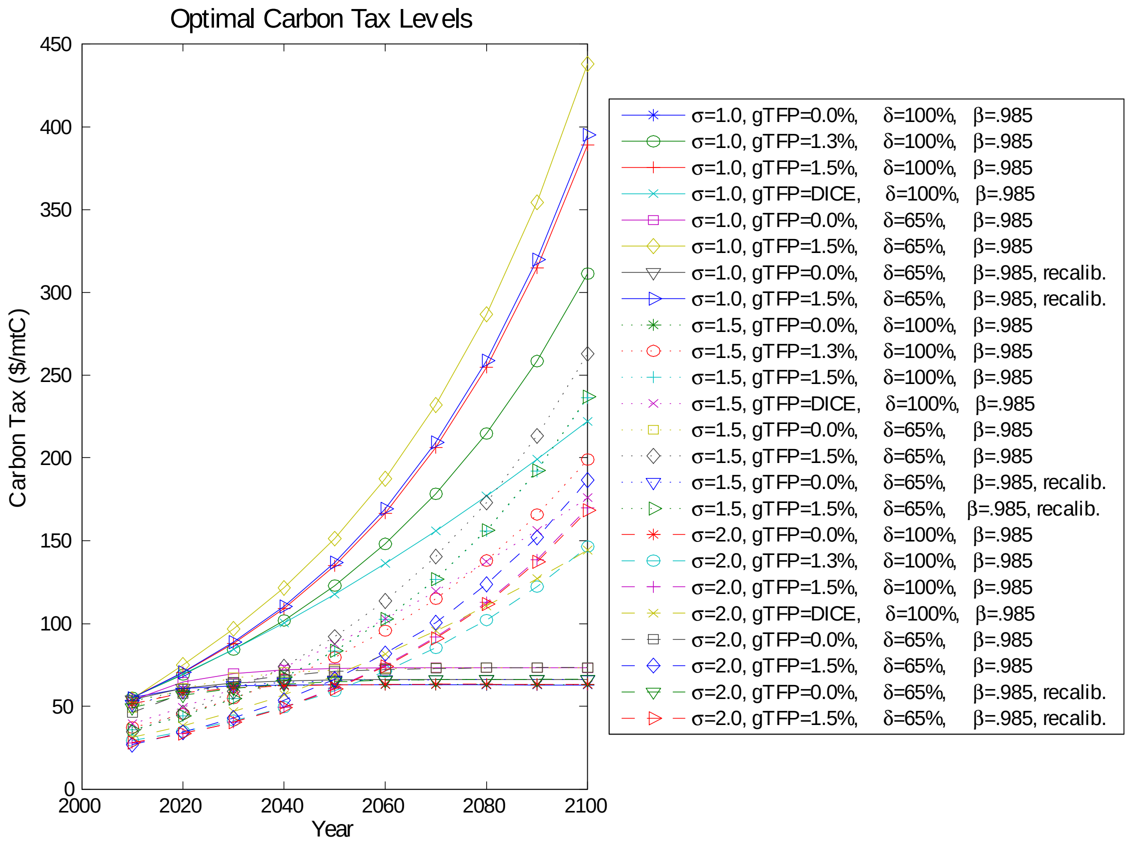


Figure 3.8

The results suggest that the optimal carbon tax level in the year 2100 is sensitive to assumptions made about the structure of preferences and TFP growth. However, the optimal carbon tax as of 2010 ranges only from \$28 to \$55/mtC (given $\beta = .985$). Figures 3.9 and 3.10 focus separately on the evolution of optimal carbon tax levels across TFP growth rates for $\sigma = 1$ and $\sigma = 2$. The results suggest that uncertainty about future TFP growth has larger implications for optimal carbon tax levels later on in the century if utility is logarithmic. Conversely, if $\sigma = 2$, we see that uncertainty about future TFP growth plays a relatively larger role in determining optimal carbon tax levels in the near future:

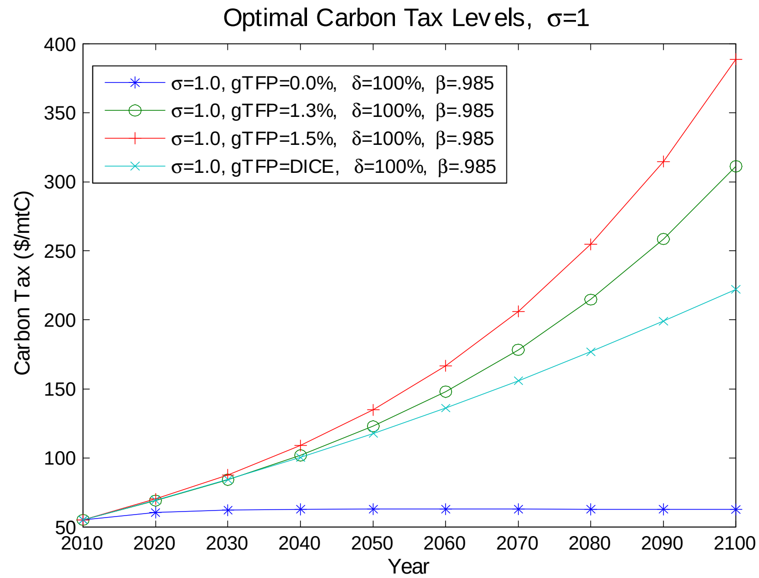


Figure 3.9

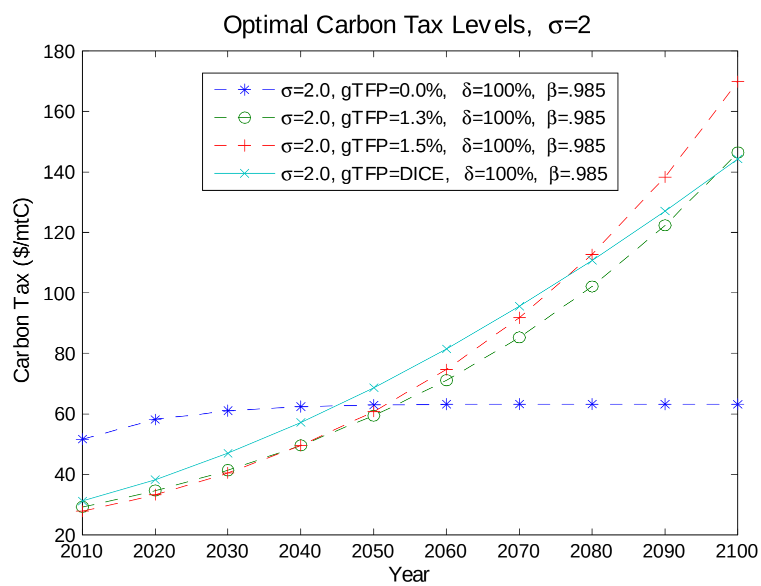


Figure 3.10

Finally, to evaluate the importance of recalibration of the initial capital stock when changing the depreciation rate, Figure 3.11 compares optimal carbon taxes with and without recalibration:

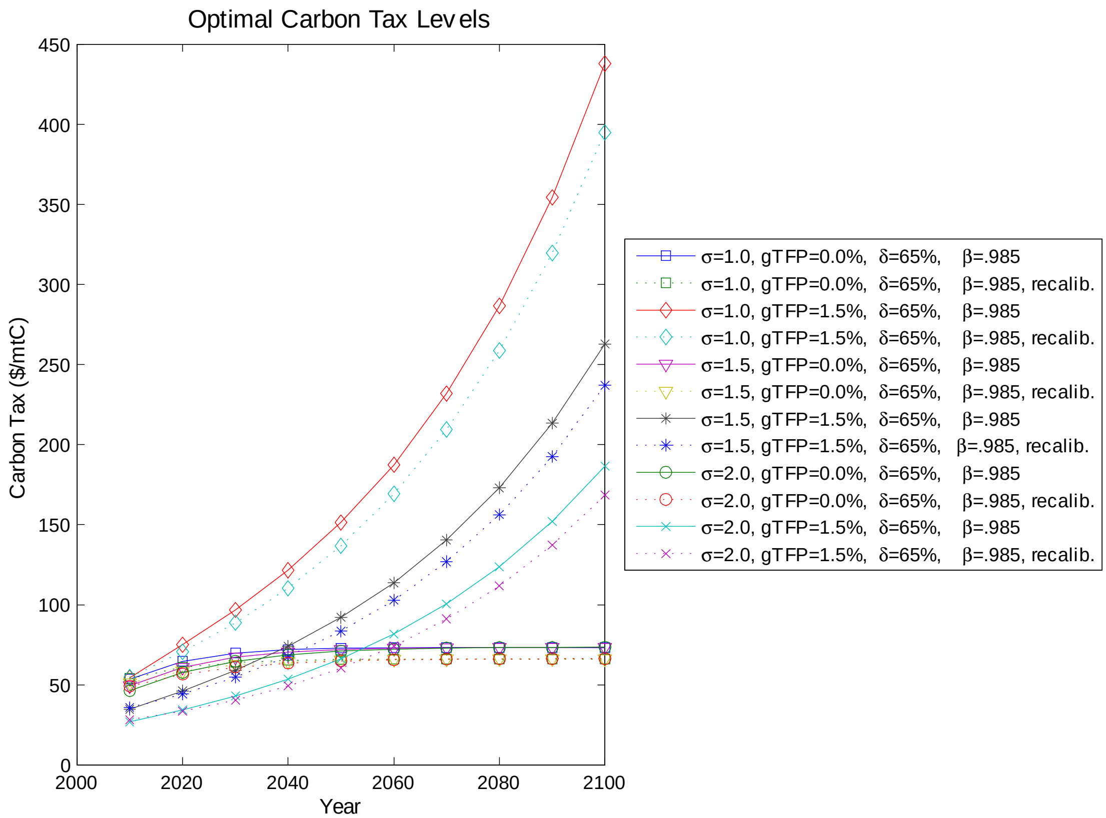


Figure 3.11

The results suggest that recalibration affects optimal carbon tax levels only slightly. In contrast, considerably larger differences arise due to changes in assumed output growth rates and utility function curvature.

4 Analytic Approximation

GHKT provide an analytic, closed-form solution for the optimal carbon tax-GDP ratio in the Benchmark case (11). In the alternative cases considered, $\widehat{\Lambda}_t$ deviates from its benchmark value for two reasons: transitional dynamics in the savings rate, and changes to effective discounting. For the functional forms considered, GHKT's general optimal carbon tax formulation (8) becomes:

$$
\begin{aligned}
\widehat{\Lambda}_t &\equiv \frac{\Lambda_t}{Y_t} = \sum_{j=0}^{\infty}\beta^j \left(\frac{C_{t+j}}{C_t}\right)^{-\sigma}\left(\frac{Y_{t+j}}{Y_t}\right)(-\gamma)\left(1-d_j\right) \\
&= \sum_{j=0}^{\infty}\beta^j \frac{\left(C_t^{\sigma}/Y_t\right)}{\left(C_{t+j}^{\sigma}/Y_{t+j}\right)}(-\gamma)\left(1-d_j\right) \qquad (31)
\end{aligned}
$$

Once the economy has reached the point where savings rates are stabilized, one can rewrite (31) using the fact that:

$$
\frac{C_{t+j}}{Y_{t+j}} = \overline{\theta} \ \forall j
$$

and hence:

$$
\frac{C_{t+j}^{\sigma}}{Y_{t+j}} = \frac{\left(Y_{t+j}\overline{\theta}\right)^{\sigma}}{Y_{t+j}} = \overline{\theta}^{\sigma} Y_{t+j}^{\sigma-1}
$$

yielding:

$$
\begin{aligned}
\widehat{\Lambda}_t &= \sum_{j=0}^{\infty}\beta^j \frac{\left(C_t^{\sigma}/Y_t\right)}{\left(C_{t+j}^{\sigma}/Y_{t+j}\right)}(-\gamma)\left(1-d_j\right) \\
&= \sum_{j=0}^{\infty}\beta^j \left(\frac{Y_t}{Y_{t+j}}\right)^{\sigma-1}(-\gamma)\left(1-d_j\right) \qquad (32)
\end{aligned}
$$

If the economy exhibits a constant growth rate, g_y, equation (32) becomes:

$$
\begin{aligned}
\widehat{\Lambda}_t &= \sum_{j=0}^{\infty}\left[\beta\left(1+g_y\right)^{1-\sigma}\right]^j \cdot (-\gamma)\left(1-d_j\right) \\
&= \gamma\left(\frac{\phi_L}{1-\beta(1+g_y)^{(1-\sigma)}} + \frac{(1-\phi_L)\phi_0}{1-(1-\phi)\beta(1+g_y)^{(1-\sigma)}}\right) \qquad (33)
\end{aligned}
$$

Expression (33) represents a slightly modified version of the GHKT benchmark formulation (11). In the benchmark case, with $\sigma = 1$, equation (33) reduces to the standard (11). When $\sigma > 1$, formulation (33) approximates the optimal carbon tax-GDP ratio, and represents it exactly if savings rates and GDP growth rates are constant.

Figure 4.1 compares actual estimates of $\widehat{\Lambda}_t$ against the corresponding approximations based on (33). All cases in Figure 4.1 assume an annual discount factor of $\beta = .985$ and full depreciation over the course of a decade ($\delta = 100\%$). Note that the long-run growth rate of *labor* productivity was used to estimate output growth g_y, as would be appropriate for an economy on a balanced growth path.[2] Given that oil inputs are decreasing over time, however, this procedure over-estimates the true long-run growth rate of the economy, which is actually gradually decreasing over time. Appendix B provides a comparison of output growth rates and corresponding labor productivity growth rates across model scenarios. Overall, however, expression (33) arguably approximates $\widehat{\Lambda}_t$ well.

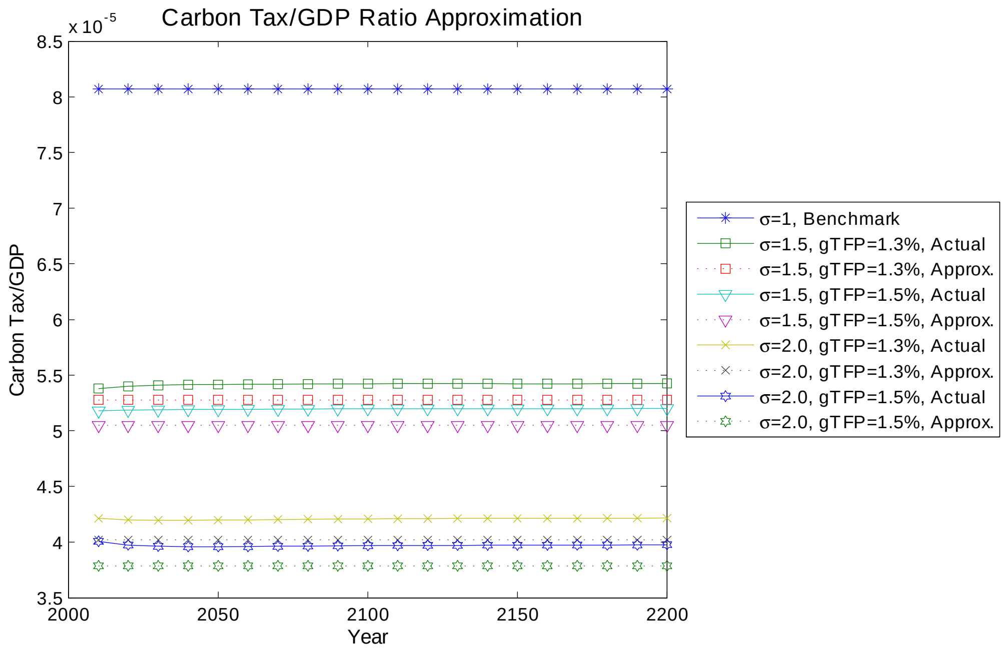


Figure 4.1

The two main shortcomings of the approximation are (i) that it does not capture transitional dynamics, and (ii) that it tends to slightly under-estimate $\widehat{\Lambda}_t$ because it over-estimates g_y. For example, for the case of $\sigma = 1.5$, $gTFP = 1.3\%$ per year, $\delta = 100\%$, and $\beta = .985$, the decadal growth factor of final good sector labor productivity is 1.2190, but the realized average output growth factor between the years 2060-2410 is only 1.1991 (see Appendix B). Figure 4.2 zooms in on this case to highlight its implications for the carbon tax-GDP ratio approximation:

2 Given the Cobb-Douglas formulation for final goods production, one can find a labor productivity growth rate g_z that is equivalent to a given TFP growth rate g_{TFP} via: $g_z = (1 + g_{TFP})^{\frac{1}{1-\alpha-v}} - 1$.

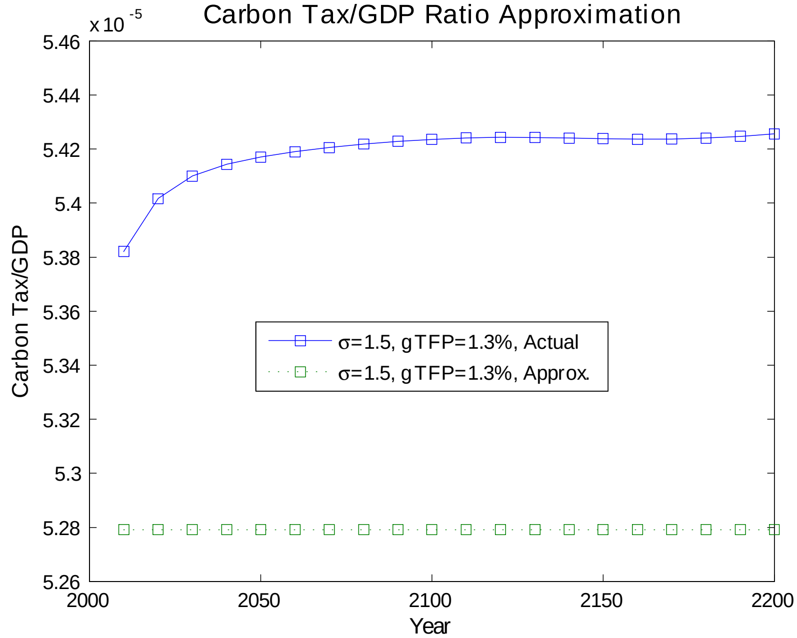


Figure 4.2

The results in Figure 4.2 suggest that both concerns surrounding the approximation (33) are of modest magnitude.

4.1 Sensitivity with Adjusted Discount Factors

The sensitivity analysis above changes the intertemporal elasticity parameter σ and the long-term growth rate of consumption whilst maintaining a constant discount factor β. Effective discount rates thus differ across the scenarios considered. This section presents an alternative sensitivity analysis that adjusts β when changing σ and g_z so as to maintain consistency with the benchmark case. More specifically, this section focuses on parameter combinations of σ, g_z, and β for which the *approximated* carbon tax-GDP ratio as defined in (33) remains at the benchmark value of $\widehat{\Lambda}_t = .0000807$. That is, for a given combination of σ and g_z, we consider what the pure rate of social time preference would have to be such that the effective discount factor remains at the benchmark value of $\beta = (.985)^{10}$:

$$\begin{aligned} \widehat{\beta}(1+g_z)^{1-\sigma} &= (.985)^{10} = \beta^{\text{Benchmark}} \\ \widehat{\beta} &= \frac{(.985)^{10}}{(1+g_z)^{1-\sigma}} \end{aligned} \tag{34}$$

Figure 4.3 displays values of $\widehat{\beta}$ that will maintain the benchmark optimal carbon tax approximation for a given combination of (σ, g_z) as per (34) in annual levels:

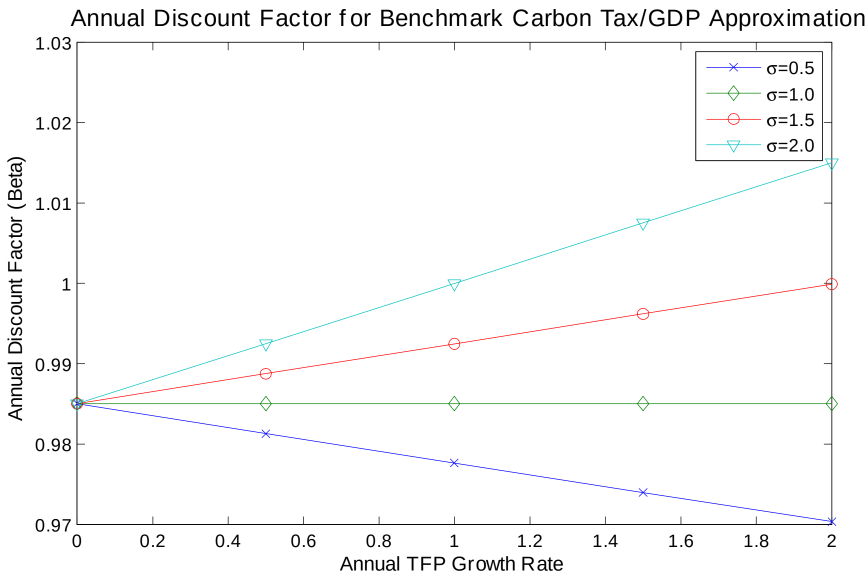


Figure 4.3

Next, Figure 4.4 displays actual and approximated optimal carbon tax-GDP ratios for a range of parameter values with β adjusted as in Figure 4.3:[3]

[3] Scenarios that would require $\beta > 1$ were not considered.

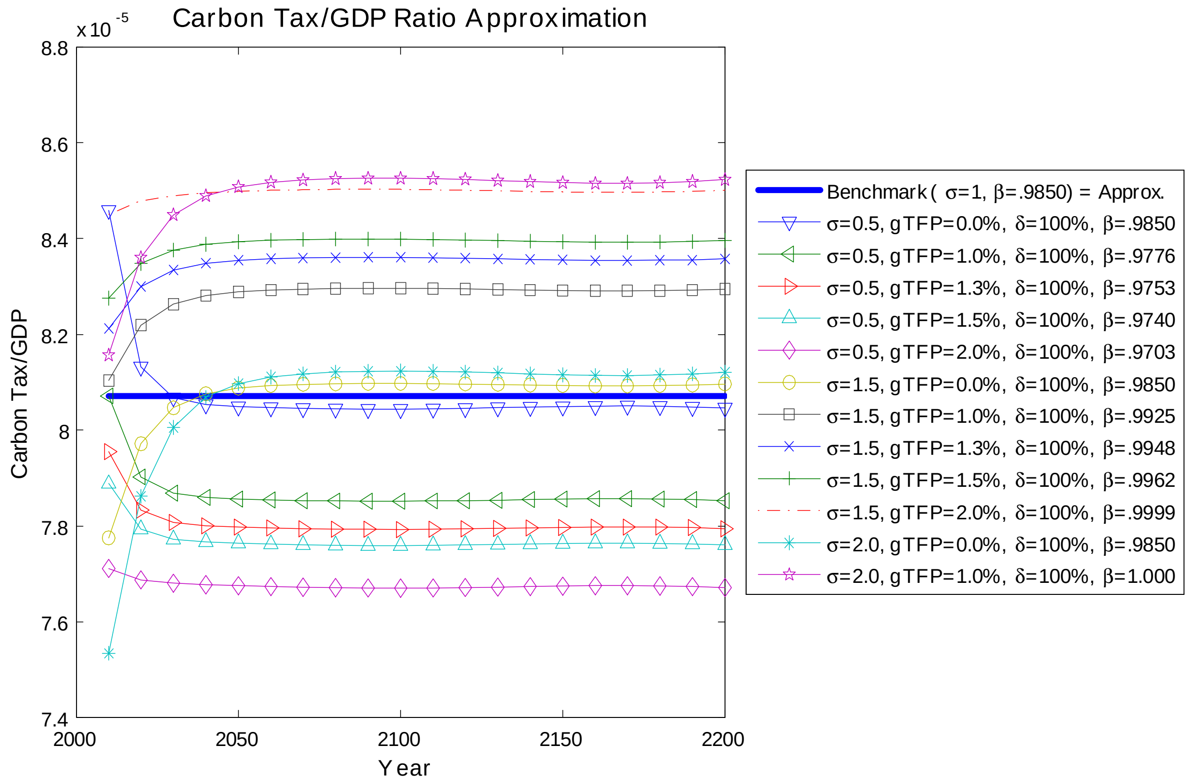


Figure 4.4

With the discount factor set as in (34), the approximated optimal carbon tax-GDP ratio is by construction identical to the benchmark value. As expected, the actual optimal carbon tax-GDP ratio thus lies close to the benchmark value in all cases considered. The actual values of $\widehat{\Lambda}_t$ deviate slightly from the approximation only for two reasons: (1) transition dynamics, and (2) the fact that the actual long-term output growth rate g_y falls short of the long-term labor productivity growth rate g_z due to declining oil inputs. For example, in the scenario $\sigma = 2$, $gTFP = 1\%$, $\delta = 100\%$, and $\beta = 1$, the average output growth factor between the years 2060-2410 is 1.1517, but the decadal labor productivity growth factor is 1.1627 (see Appendix B).

Finally, to put the results from Figure 4.4 in further perspective, Figure 4.5 compares actual and approximated carbon tax-GDP ratios in the case where $\sigma = 1.5$, both with and without adjustments to β to keep effective discount rates close to the benchmark:

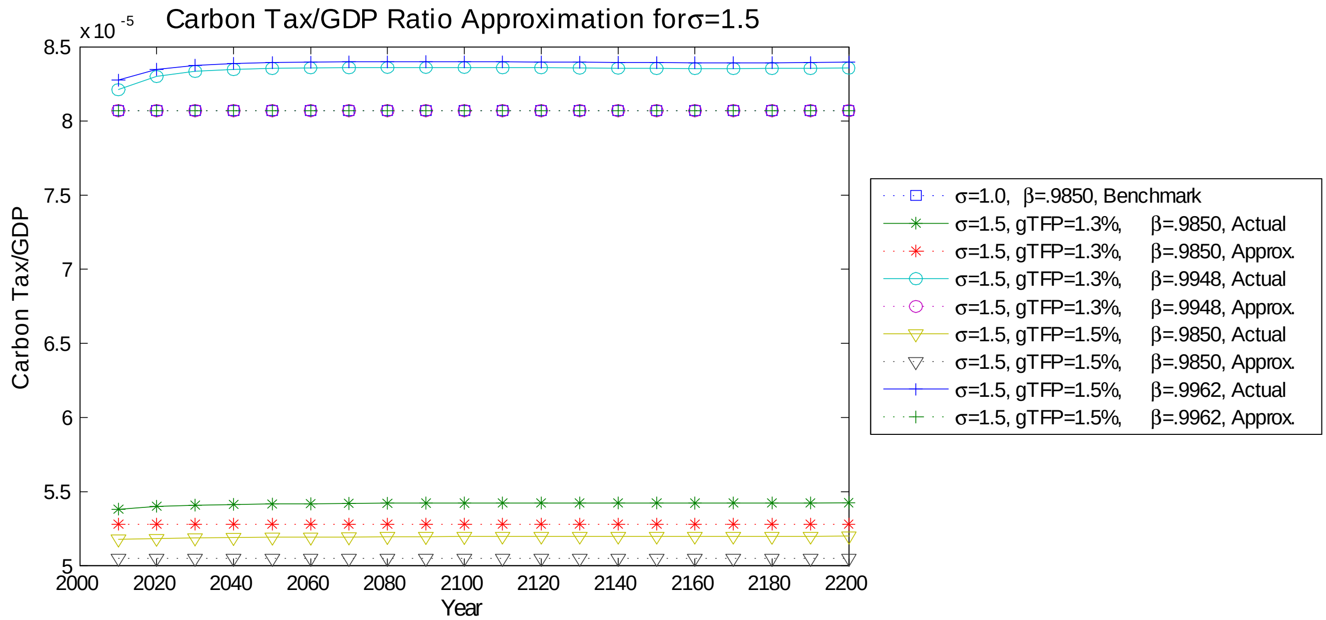
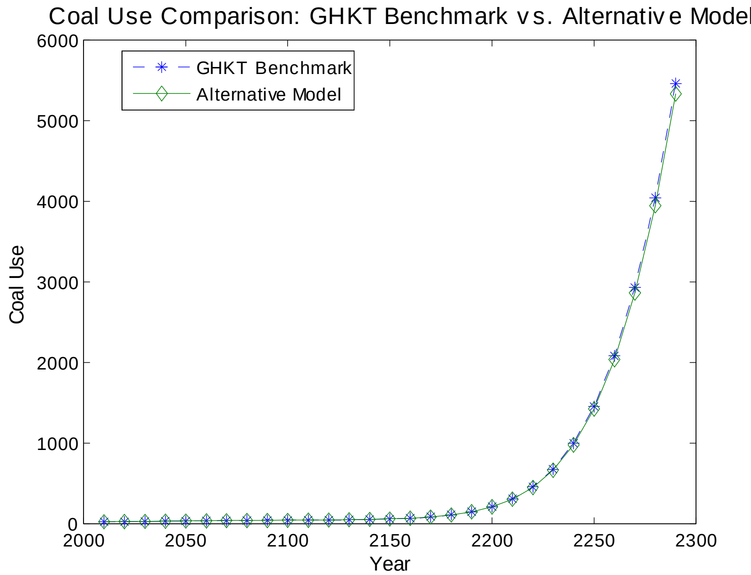


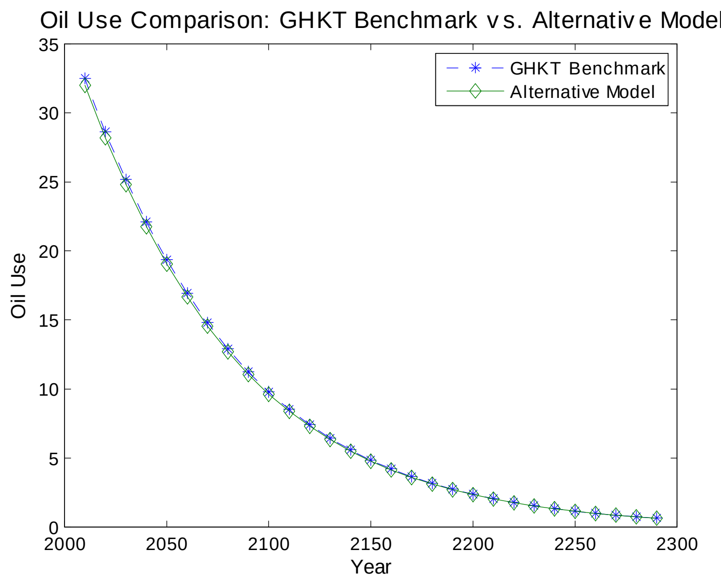
Figure 4.5

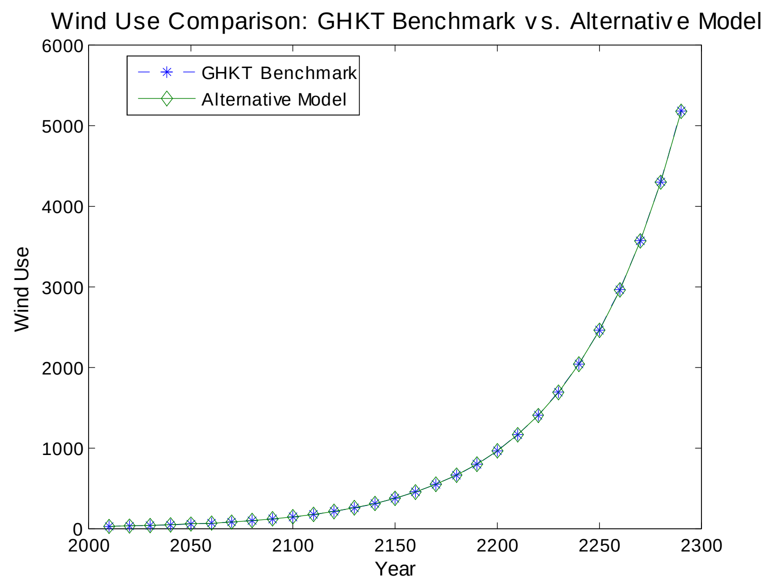
As expected, the optimal carbon tax-GDP ratio remains close to the benchmark value and its approximation when the discount factor is adjusted along with σ and g_z. In contrast, bigger deviations from the benchmark occur when σ and g_z are changed and β is held constant. However, even in those cases, the modified optimal carbon tax formulation (33) arguably captures the optimal carbon tax-GDP ratio well.

5 Appendix A: Matching GHKT's Benchmark Quantitative Results

This appendix compares quantitative results for energy inputs in the benchmark case ($\sigma = 1, \delta = 1, gTFP = 0$) obtained by GHKT to those obtained by the alternative numerical model used throughout this note. The alternative Matlab model replicates GHKT's results for the benchmark case, as desired:


Coal Use Comparison: GHKT Benchmark vs. Alternative Model
GHKT Benchmark
Alternative Model
Coal Use
Year
6000
5000
4000
3000
2000
1000
0
2000
2050
2100
2150
2200
2250
2300


Oil Use Comparison: GHKT Benchmark vs. Alternative Model
GHKT Benchmark
Alternative Model
Oil Use
Year
35
30
25
20
15
10
5
0
2000
2050
2100
2150
2200
2250
2300



6 Appendix B: Output Growth Factors

This appendix details three measures of output growth factors for the central model scenarios: Averaged across all periods between years zero and 400 ($\overline{G_y}_{t=0}^{t=400}$) (calendar years 2010-2410), between years 50 and 400 ($\overline{G_y}_{t=50}^{t=400}$), and in the decade between 2110 and 2120 ($G_{y,100}$). Excluding the early periods helps provide a cleaner comparison between actual output growth and labor productivity growth rates in the final goods production sector, eliminating differences in growth rates due to transitional dynamics early on. Labor productivity in the energy production sector is assumed to grow at 2% per year in all scenarios.

$G_z^{dec} = 1.000$ $\overline{G_y}_{t=0}^{t=400}$		β					
		.985			.990	.995	.999
		$\delta = 1$	$\delta = .65$, no Rec.	$\delta = .65$, Rec.			
	0.5	1.0023					
σ	1	1.0022	1.0064	1.0035	1.0047	1.0070	1.0089
	1.5	1.0022	1.0062	1.0035	1.0046	1.0069	1.0088
	2	1.0022	1.0062	1.0034	1.0046	1.0068	1.0086

Table B1: Average growth factors between years 0-400 for 0% TFP growth

$G_z^{dec} = 1.000$ $\overline{G_y}_{t=50}^{t=400}$		β					
		.985			.990	.995	.999
		$\delta = 1$	$\delta = .65$, no R.	$\delta = .65$, R			
	0.5	0.9986					
σ	1	0.9986	0.9989	0.9988	1.0007	1.0027	1.0048
	1.5	0.9987	0.9991	0.9989	1.0007	1.0027	1.0047
	2	0.9988	0.9993	0.9991	1.0008	1.0027	1.0046

Table B2: Average growth factors between years 50-400 for 0% TFP growth

$G_z^{dec} = 1.000$ $G_{y,100}$		β					
		.985			.990	.995	.999
		$\delta = 1$	$\delta = .65$, no R.	$\delta = .65$, R			
	0.5	0.9986					
σ	1	0.9987	0.9988	0.9988	1.0005	1.0021	1.0033
	1.5	0.9988	0.9990	0.9989	1.0005	1.0021	1.0032
	2	0.9988	0.9993	0.9992	1.0005	1.0021	1.0032

Table B3: Growth factor between 2110-2120 for 0% TFP growth

$G_z^{dec} = 1.2190$ $\overline{G_y}_{t=0}^{t=400}$		β					
		.985	.990	.995	.999	.9753	.9948
	0.5	1.2108				1.2051	
σ	1	1.2054	1.2084	1.2112	1.2134		
	1.5	1.1999	1.2030	1.2059	1.2082		1.2058
	2	1.1945	1.1975	1.2005	1.2029		

Table B4: Average growth factors between years 0-400 for 2% labor productivity growth

$G_z^{dec} = 1.2190$ $\overline{G_y}_{t=50}^{t=400}$		β					
		.985	.990	.995	.999	.9753	.9948
	0.5	1.2082				1.2033	
σ	1	1.2036	1.2061	1.2085	1.2110		
	1.5	1.1991	1.2016	1.2041	1.2060		1.2040
	2	1.1945	1.1971	1.1996	1.2016		

Table B5: Average growth factors between years 50-400 for 2% labor productivity growth

$G_z^{dec} = 1.2190$		β					
$G_{y,100}$		.985	.990	.995	.999	.9753	.9948
	0.5	1.2076				1.2035	
σ	1	1.2038	1.2058	1.2078	1.2092		
	1.5	1.2000	1.2021	1.2041	1.2057		1.2039
	2	1.1962	1.1983	1.2004	1.2021		

Table B6: Growth factor between 2110-2120 for 2% labor productivity growth

${G^{dec}}_z = 1.2531$		β				
$\overline{G_y}_{t=0}^{t=400}$		.985			.974	.9962
		$\delta = 1$	$\delta = .65$, no R.	$\delta = .65$, R		
	0.5				1.2364	
σ	1	1.2368	1.2406	1.2373		
	1.5	1.2304	1.2337	1.2305		1.2372
	2	1.2240	1.2269	1.2238		

Table B7: Average growth factors between years 0-400 for 1.5% TFP growth

${G^{dec}}_z = 1.2531$		β				
$\overline{G_y}_{t=50}^{t=400}$		.985			.974	.9962
		$\delta = 1$	$\delta = .65$, no R.	$\delta = .65$, R		
	0.5				1.2349	
σ	1	1.2353	1.2354	1.2354		
	1.5	1.2299	1.2301	1.2300		1.2357
	2	1.2245	1.2247	1.2245		

Table B8: Average growth factors between years 50-400 for 1.5% TFP growth

${G^{dec}}_z = 1.2531$		β				
$G_{y,100}$		.985			.974	.9962
		$\delta = 1$	$\delta = .65$, no R.	$\delta = .65$, R		
	0.5				1.2351	
σ	1	1.2354	1.2355	1.2355		
	1.5	1.2310	1.2311	1.2310		1.2357
	2	1.2266	1.2267	1.2266		

Table B9: Growth factor between 2110-2120 for 1.5% TFP growth

$G_z^{dec} = 1.1627$	$\overline{G}_{y_{t=0}}^{t=400}$	$\overline{G}_{y_{t=50}}^{t=400}$	$G_{y,100}$
$(\sigma = 0.5,\ \beta = .9776)$	1.1532	1.1509	1.1510
$(\sigma = 1.5,\ \beta = .9925)$	1.1537	1.1514	1.1515
$(\sigma = 2.0,\ \beta = 1.000)$	1.1540	1.1517	1.1517

Table B10: Growth factors for 1% TFP growth

$G_z^{decade} = 1.3499$	$\overline{G}_{y_{t=0}}^{t=400}$	$\overline{G}_{y_{t=50}}^{t=400}$	$G_{y,100}$
$(\sigma = 0.5,\ \beta = .9703)$	1.3251	1.3246	1.3248
$(\sigma = 1.5,\ \beta = .9999)$	1.3263	1.3257	1.3257

Table B11: Growth factors for 2% TFP growth

References

[1] Golosov, Mikhail, John Hassler, Per Krusell, and Aleh Tsyvinski (2013) "Optimal taxes on fossil fuel in general equilibrium" Working paper.

[2] Nordhaus, William D. (2010) *DICE-2010 model*, available at http://nordhaus.econ.yale.edu/RICEmodels.htm